NOTE TO FILE
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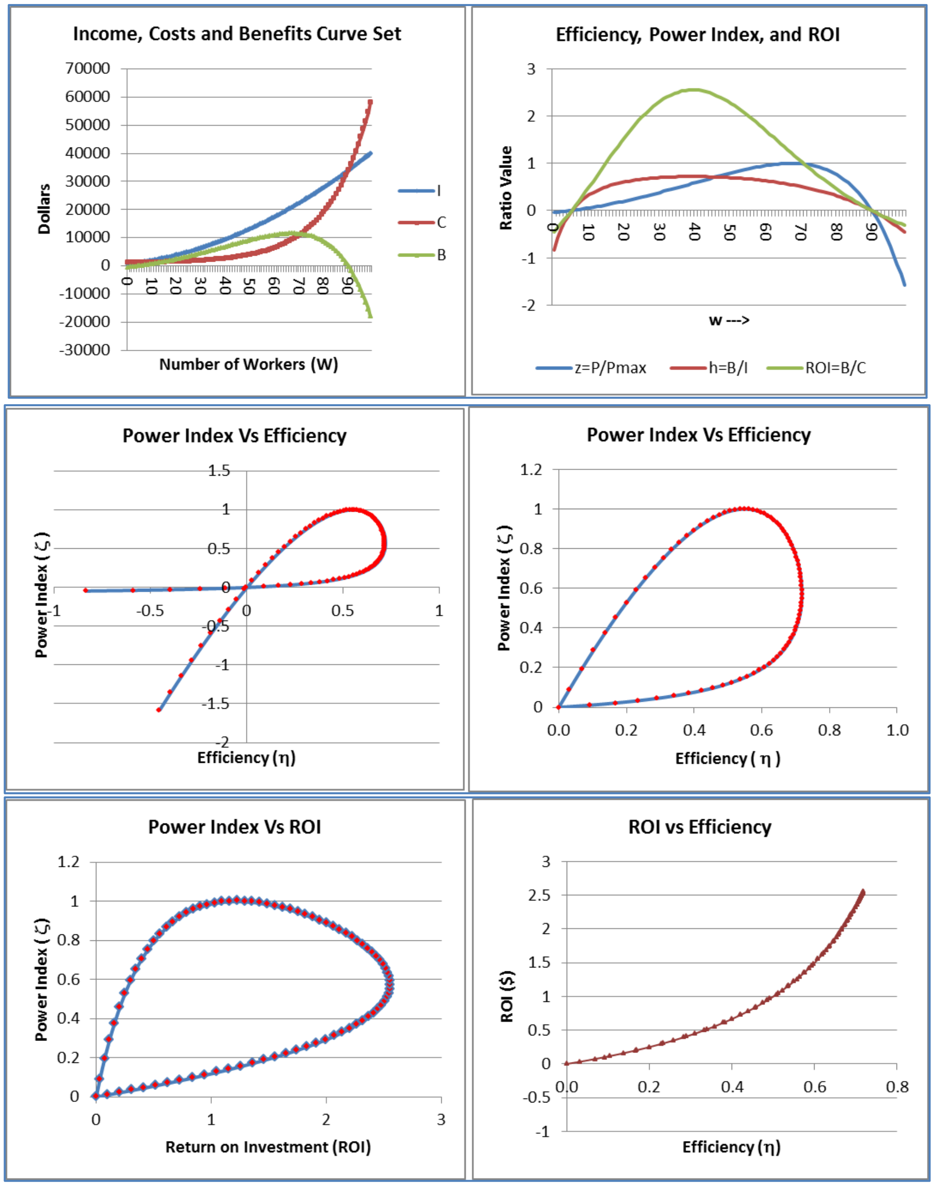
Income, Costs, Benefits, Time and Power-Efficiency

Frontispiece



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K. 25th Oct 2015, "The Persistence of Galbraith", Hector Mercadier, http://themarketmogul.com/persistence-galbraith
L. 151224 XLS ICBT and PowEff R6.xlsx
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2 Purpose

To progress my understanding of the proposed ubiquity of the power-efficiency curves that appear in the works of H. T. Odum. My intent in this particular note is to explore the mathematics of such curves in association with the concepts and scenarios described in the Ref J NTF.

3 Background

Much of this background story is copied from the Ref J NTF. This NTF continues that particular line of thought, but in a somewhat different direction, and using a different technique of analysis. So, this is NOT Part II of the study that I foresaw when I wrote Part I at Ref J.

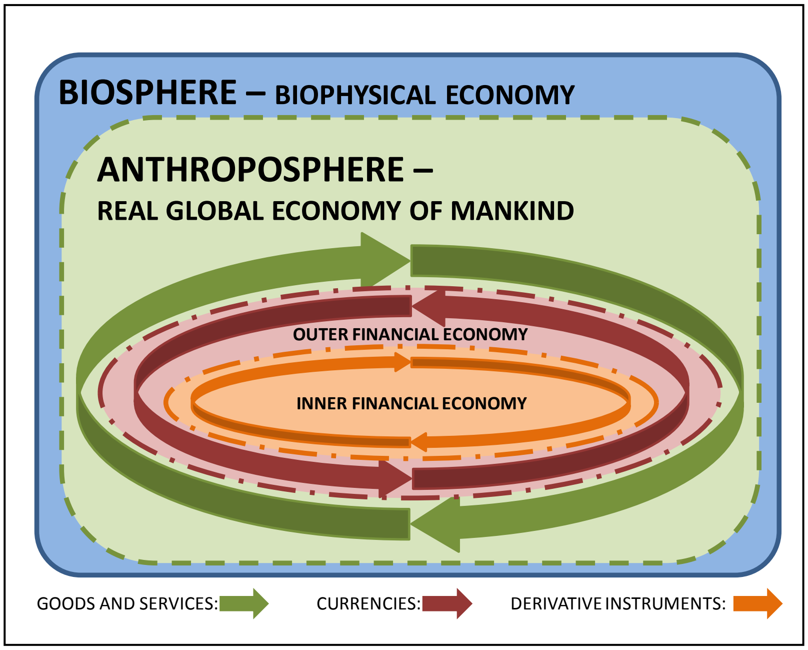
In my various NTFs I have referred to a set of relationships between the power and the efficiency of a system as (a) power-efficiency curves; (b) concave downwards unit maps; (c) CCD curve (concave downwards); (d) Goldilocks curves (not too hot, not too cold, just right); and, perhaps other names. While working on this note I have had the startling realization that power-efficiency relations are not all functions, and are not all CCD. So, I am going to avoid the use of monikers b and c, and try to stick with a and d.

I am trying to show that the maximum power principle (or MPP) is ubiquitous in all persistent forms of economic business activity that regularly produce profits. I am using Odum's definitions of biophysical power and efficiency, as he expressed them in his discussion of Atwood's Machine (see Refs A-D) when considering flows of energy, to find analogous definitions of financial power and efficiency, when considering flows of capital (see Refs E-H).

I need to make a careful and intentional distinction:

- I strongly believe that, IF the MPP is ubiquitous in persistent thermodynamic systems, biophysical systems and ecosystems, then it is also ubiquitous in economic systems. This conclusion is by mere reason of the fact that an economic system is, de facto, a thermodynamic engine of sorts. I say "IF" in capitals, because its ubiquity in thermodynamic systems has NOT been proven, or even empirically researched, as far as I know, though I think there is a strong logical argument that it MUST be ubiquitous. So, there are three levels of my "beliefs" here that have not been proven.

- I believe it applies to all thermodynamic, biological and ecological systems; and
- I believe it also applies to the real biophysical (or thermodynamic) layer of economies. In my ModEco model I call this the lower layer, so that's what I will call it here as well.
- However, in recent years we have seen a phenomenon which is peculiar in the extreme. The upper layer of many national economies and of the global economic system as a whole – that layer in which such esoteric and insubstantial things such as guarantees, warranties, insurances, currencies, commodity risks, and investment risks are bought and sold – in that uppermost layer, the economy is much less a thermodynamic engine, and more of a logical mathematical engine. This logical system has in many ways broken free of biophysical constraints, and functions on flows of disembodied currencies that exist only as flows and stores of digits in a global network of computers. This is a self-organizing and persistent sub-system of the real lower-level economy mentioned above, and, in its own right, must have an auto-catalyzing process that makes it self-organizing and persistent.



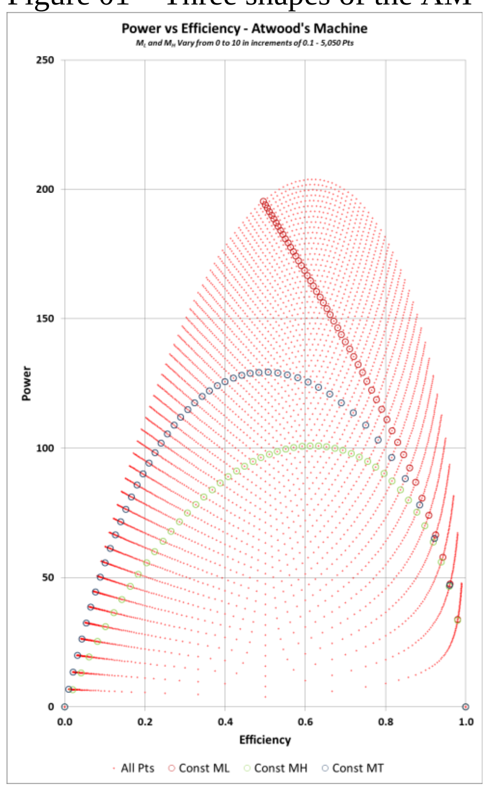
In my ModEco documentation I refer to the upper and lower layers of the economy, but I am coming to think there are four nested economies, and a more accurate analogy for the financial part might be outer and inner respectively. The biosphere is the complete economy of the Earth. Mankind's global economy is within that, and is the "real" economy – that portion of the biosphere we now call the anthroposphere, which is a sub-economy – a subsystem. The "real" economy consists of flows and stores of matter and energy, converted to goods and services. The outer financial economy facilitates the real economy, and exists in mirror image of the real economy, in that all transfers of mass and/or energy in one direction are mirrored in the outer financial economy by transfers of currency in the other direction. This financial economy also seems to have two subsystems, as discussed above. The outer portion of the financial system in which currency (of some sort) is exchanged for biophysical goods and services I consider the financial part of the real economy. But the portion in which currencies are exchanged for other non-tangible currencies, or risk-based assets, or for the management and sale of such assets, that portion I consider to be a "logical" financial economy. It is contained within the inner part of the "real" economy, and whether a particular transaction belongs to one or the other financial subsystem may be a matter of judgment. I suppose it is a matter of degree of loosening the constraints of reality, and not absolute breakage of the linkage with reality. Even the sale of the most abstract derivative instrument requires the consumption of some small amount of electricity as EFT (electronic funds transfer) protocols are executed by computers, and bytes are altered in distant computer banks, so it is not totally de-linked from the real system. So, a rule of thumb to draw a line between the two might be this: If no organism or biophysical machine other than computers expended energy to complete the transaction, then it is wholly within the logical financial system. So, whether I call this the inner financial system, or the upper financial system, it is the logical financial system.

So, my goal in these mathematical studies is to show that **something like the MPP** (a non-thermodynamic but logical extension of the concepts of the MPP) is an ubiquitous agent causing economic self-organization in the capital flows of the lower economy, but, more importantly, also enabling the self-organization of the upper level of the financial system as well.

I am hoping this particular note and exercise, and, perhaps others to follow, will help me to break through the mental blockage I have hit on my Ref E mathematical paper on the MPP. My mind gets tangled in three or four kinds of problems:

- Can you always find an analytic expression relating power and efficiency? It seems not. So any proof like I was attempting at Ref E must use logic that avoids specific analytic equations, but depends on the characteristics of imagined or presumed analytic relationships, in some fashion.
- What are the proper analogies between energy flows in the AM and capital flows in an economy? I have come to think that these three analogies are valid:
  - The total high-grade energy input (the gravitational potential energy in the large mass) in the AM is analogous to the total income resulting from an economic activity.
  - The energy that is converted to kinetic energy, and then expended as waste, in the AM is analogous to the costs of production associated with an economic activity.
  - The high-grade energy that is transferred (as gravitational potential energy in the small mass) in the AM is analogous to the benefits or profits that arise from the economic activity, that are then passed on to the next iteration of business activity.
- Are all capital flows and stores reducible to thermodynamic flows and stores, or do they have additional power and effect beyond their use as biophysical proxies. E.g. a dollar is a lien on future production. When the total value of liens exceeds the total annual production for many years, this idea loses its meaning. And yet, dollars and debts pile up in a self-creating persistent fashion. I would guess that the laws of thermodynamics would have very little impact when the linkage between the upper layer of financial economy and reality is so weak and tenuous, but, nevertheless, there appears to be a strong influence towards self-organization in the upper layer. Perhaps the influence is not "because of" the thermodynamic connection, but "just like" in thermodynamics. What is it that influences both thermodynamic engines and logical engines to self-organize?

Figure 01 – Three shapes of the AM



- How are the characteristics of quality of product and cost related? In the language of information theory, the more time a person spends perfecting a product, the greater is the information content, and the less is the informational entropy content. So, there is NOT just a thermodynamic "entropy tax" when a product is improved, there is also a peculiar dynamic by which internal informational entropy associated with that product is reduced, even as the external informational and thermodynamic entropy rises. The expenditure of the workers thermodynamic energy causes a reduction in internal informational entropy. This is a form of autocatalytic effect that crosses the boundary between the real economy and the logical economy.

Here is a brief explanation of the files mentioned in the References section.

- Ref A is a NTF in which I try to understand how Atwood's Machine (the AM) could provide insight into the maximum power principle. It is a naive item, first written in Sep 2014, and revised several times, but that is where I started. It, in fact, does little to address the MPP itself, but focuses on the Newtonian mechanics surrounding the AM.
- Ref B is a NTF in which I look closely at the power-efficiency function that comes out of the Ref A study of the AM. This was essentially a mathematical study of the characteristics of the very strange function which was my first introduction to these Goldilocks curves.
- In Ref C I revisit the AM, but this time with a focus on the connection to the MPP. This is where my first real insight into the MPP starts to develop, greatly inspired by comments from Charles Hall, Dan Campbell and Sholto Maud.
- In the Ref D NTF I examine closely how the three different power-efficiency curves arise from the dynamics of the AM, and figure out a way to show them all on a single graphic. To do this I had to make four independent scatter graphs on the same graph. It's cool! (See Figure 01.) I have been able to build two agent-based models that demonstrate persistent evolutionary systems that evolve to maximum power for each of the two concave-downwards (CCD) curves shown in Figure 01. Those models that followed the non-concave curve evolved and ultimately collapsed, and were not persistent.
- At the Ref E PPR I attempted to produce a general mathematical argument as to why the MPP is active everywhere, and, in particular, in the upper layer or financial portion of economic systems. This is a VERY DRAFT document that is seriously flawed, and I don't know yet how to fix it. However, I am working on that, and that is why I undertook this exercise.
- In the Ref F paper I took the first significant step to resolving how to fix the Ref E paper by restating the Lotka-Odum MPP as three extremely general and falsifiable hypotheses. This was helpful, as I came to realize that the Ref E paper is only addressing the second part of the three parts of the MPP, trying to show the ubiquity of CCD curves.
- Ref G is my presentation powerpoint slide deck used to explain the Ref F paper at CANSEE 2015, in the BPE track.
- Ref H is the MP4 video that I used to demonstrate the MPP in action during the Ref G presentation, based on my assumptions in the Ref F paper, and using an agent-based computer model called MppLab I.

My demonstration of the MPP at Refs G and H rests on the **assumption** in Ref F that hypothesis MPP#2 is correct. I.e. that the hump-backed (i.e. CCD) power-efficiency curves are ubiquitous in all persistent systems. But, now I need to show that this can be true. And that is what I was trying to do, in some sense, in the Ref E paper. And that is what I am trying to do now, in a scaled down fashion, in this and subsequent notes.

Refs I and J are the first part of a study of a constant-demand workshop which I did not finish, and which is the immediate predecessor of this exercise. I got a better idea about how to analyze the cases or scenarios.

Ref K is an article about Galbraith's ideas that seems pertinent to the MPP.

Refs L and M are the files in which the graphics for this note were developed.

So, that's a quick intellectual history of how and why I got to be here.

4 Discussion

4.1 Definitions and boundary conditions

4.1.1 The Seven Primary Curve Types

Let I(w), C(w) and T(w) be smooth curves defined on the interval $w \in [0, \infty]$ where:

- I(w) represents income from a batch of widgets sold.
- C(w) represents the costs associated with making that batch of widgets.
- T(w) represents the duration of time required to make the batch of widgets.

Let B(w), η(w), P(w) and R(w) be four additional curves, derived from the first three as follows:

- B(w) represents the profits, or benefits, that derive directly from the expenditures C(w), calculated as B(w)=I(w)-C(w). It, in turn, is used to calculate the next three business indicator ratios.
- P(w) represents the power of the benefits, calculated as P(w)=B(w)/T(w). I also use something I call a "Power Index" which is a normalized data series calculated as $\zeta(w)=P(w)/P_{max}$, where P_{max} is the maximum value of P(w) on the interval $w \in [0, \infty]$. For my purposes, P(w) and ζ(w) are interchangeable since I am interested in shapes of curves, and these two curves have similar shape and impact.
- η(w) represents efficiency, calculated as B(w)/I(w).
- R(w) represents the return on investments and is calculated as R(w)=B(w)/C(w).

4.1.2 It's about time

Together, for some business, these I(w), C(w) and B(w) curves represent the income, costs and profits associated with some control variable (w) such as, but not limited to, the number of workers hired, for example, to make the batch of widgets within a time duration T(w), which is also a function of w. In the AM there is a complex relationship between the time required for the process to complete (the "drop time") and the efficiency of the process. That is a BIG issue that I have chosen NOT to include in this analysis, just because there are already too many complications. I think that, because I am doing this at a high level, above the level of detail that I tried to put into the Ref J NTF, it will not damage my analysis. But I am not certain of that. So, for the purposes of this exercise, I will assume that T(w) is in fact constant over the interval $w \in [0, \infty]$, and can be denoted as T. When I use it, I note that I am assuming it is "well-behaved", though I admit I am uncertain as to exactly what that means, at this point.

4.1.3 Conditions of Inclusion

This, then, is a first background assumption, or constraint, or boundary condition, on my solutions described herein. For internal purposes of reference, I am enumerating all of my assumptions of this kind.

$T(w) = positive\ constant,$ sometimes denoted as just T	Cond 00

Define B(w) as:

$B(w) = I(w) - C(w)$	Equ 01

where B(w) are the benefits associated with the sale of that batch of widgets. In other words, B(w) represents the profits, as a function of w.

Just as there are restrictions on T(w), there are other constraints to be noted. Consistent with the idea that these income and cost numbers are associated with a persistent profitable business model, I(w) and C(w) are always positive or zero. Also, I want to rule out those cases in which income or costs grow infinitely as $w \to \infty$, since that strikes me as totally unrealistic. I cannot imagine any instance in which arbitrarily large (or infinite) income or costs are made possible by allowing some parameter to rise to infinity. So there are three more constraints for this exercise, for this NTF:

$\forall w \in [0, \infty]$: $[0 \le I(w) \le I_{max}]$ for some finite value I_{max}.	Cond 01
$\forall w \in [0, \infty]$: $[0 \le C(w) \le C_{max}]$ for some finite value C_{max}.	Cond 02
$\forall w \in [0, \infty]$: $[-C_{max} \le B(w) \le I_{max}]$.	Cond 03

The third condition is, of course, a logical consequence of the previous two conditions and the definition of B(w), and so is redundant, but I write it here for possible reference later. In this NTF I am working with intervals a lot, and I use the notation (lower bound, upper bound) to indicate an interval on the real line, or of natural numbers. Round parentheses indicate exclusion of the end point. Square parentheses indicates inclusion of the end points. In the above conditions, the first use of parentheses are examples of this, but the second use is just for parenthetical grouping. I think I have been careful to use parentheses consistently and with care for their meanings throughout the entire NTF. Watch for it on re-reading.

4.1.4 The sets of curves $\mathbb{G}$, $\mathbf{P}$ and Π

At this point I have to ask myself just what these constraints are doing! In my mind I imagine that there is some grand set $\mathbb{G}$ of smooth (differentiable) functions defined on the interval [0, ∞]. Then there is some other grand set of ordered 4-tuples of elements $G \in \mathbb{G}$ that I can denote as $\mathbf{P}$. Each element $P \in \mathbf{P}$ can be interpreted as (G_1, G_2, G_3, G_4) = (I(w), C(w), B(w), T(w)), where the position in the 4-tuple determines the role of the function in this analysis. I refer to each 4-tuple "P" as an "ICBT Curve Set". Denote the set of all allowed curve sets as Π. Then $\Pi \subset \mathbf{P}$, and the conditions and constraints are meant to be rules for determining which curve sets in $\mathbf{P}$ are allowed to be included in Π.

For example, Cond 00 is a constraint on what functions in can be allowed as T(w). I know that constraint is far too simple to be real, but I have made it simple so I can complete the analysis. I suppose I will need to come back later and loosen that very tight constraint. Also, Equ 01 is a constraint on what functions can be allowed for B(w), given I(w) and C(w). Conds 01-02 are constraints on what function can be allowed for I(w) and C(w), and Cond 03 is a further constraint on B(w), and, by implication, on I(w) and C(w). My intention in stating these constraints is to winnow out and exclude those curve sets that do not have any association with the real-world concept of profits in a persistent business activity. Admittedly, the resulting set of included ICBT curve sets would still be infinitely large. But I can nevertheless group that set of included ICBT curve sets into cases based on common characteristics, and analyze them case by case (i.e characteristic by characteristic). Then, for any given curve set, I can determine which characteristics it has, and figure out how it should behave.

4.1.5 More Conditions of Inclusion

So, continuing my list of exclusions and inclusions, i.e. my list of constraints, I also want to rule out those curve sets for which positive profits are not possible whatever the value of w. So, I exclude those curve sets for which it is true that
$\forall w \in [0, \infty]: [-C_{max} \le B(w) \le 0]$. Which means, I include only those curve sets for which:

$For\ Some\ w \in [0, \infty]: [0 \le B(w) \le I_{max}]$	Cond 04

I am unsure whether T also needs to be bounded in this fashion. Certainly it must be true that $\forall w \in [0, \infty]: T(w) > 0$, but it is part of the intrigue of the Atwood's Machine that some aspects of the energy transfer happen in association with infinite time, at perfect efficiency. Certainly, I can imagine that it might take forever to make an absolutely perfect widget, so, it makes sense to me that T(w) has a lower bound of zero, but no upper bound. The difficulty that I have some trouble with is this: the "perfect widget" is less associated with number of workers (w) and more with quality of product (say, q), a different variable. So, perhaps T should be written as T(w,q) where q is some other control variable independent of w. I will carry this somewhat clumsy notation for now, until I can sort out how to resolve the issue. Certainly, if w is number of workers, and q is quality level, then for some constant q, T should fall as w increases, and for some constant w, then T should rise as q rises, so I might guess that $T(w) \propto q^a/w^b$ for some w, q, a and b ≥ 1. But, I think it is best if I keep the notation T(w, q) for now, and sort it out later.

I then have this weak boundary condition on T(w, q):

$\forall w \in [0, \infty]: [0 \le T(w, q)]$	Cond 05

At some time in the future when I relax Cond 00, then this Cond 05 might kick in, but, for now, Cond 00 is a much tighter constraint, admitting far fewer curve sets, forming a small subset of those admitted by this constraint, making this constraint redundant. But, I leave it here for now.

Then, there are the two standard definitions coming out of consideration of the Power-Efficiency curves of the MPP. For each ICBT curve set in **Π** I can define three more functions as follows:

$P(w) \equiv B(w)/T(w)$	Equ 02
$\eta(w) \equiv B(w)/I(w)$	Equ 03
$R(w) \equiv B(w)/C(w)$	Equ 04

Where P(w) is the power (of the profits) measured in dollars per unit time, and η(w) is the efficiency of the profit-making process. These are both intentionally calculated in a fashion analogous to the calculation of efficiency for the Atwood machine. R(w) represents the concept of return on investment (or ROI), the maximization of which is a key concept in economic literature.

4.2 Rules for Using Limits

The elements of an ICBT curve set are all differentiable. That was an initial characteristic of the grand set $\mathfrak{G}$ from which all of the curves are selected. I need this to be able to use L'Hôpital's rule for evaluating indeterminate limits. I use limits extensively in what follows, so, for reference, here are the seven rules for limits that I found online. k is a constant.

$\lim_{w \to c} k = k$	Rule 01
$\lim_{w \to c} kf(w) = k \lim_{w \to c} f(w)$	Rule 02
$\lim_{w \to c}[f(w) \pm g(w)] = \lim_{w \to c} f(w) \pm \lim_{w \to c} g(w)$	Rule 03
$\lim_{w \to c}[f(w) \times g(w)] = \lim_{w \to c} f(w) \times \lim_{w \to c} g(w)$	Rule 04
$\lim_{w \to c}[f(w)/g(w)] = \lim_{w \to c} f(w)/ \lim_{w \to c} g(w)$	Rule 05
$\lim_{w \to c}[f(w)^n] = \left[\lim_{w \to c} f(w)\right]^n$	Rule 06
$\lim_{w \to c} f(w) = f(w);$ if f(w)is continuous around c.	Rule 07

5 Evaluations – Case by Case

Now, I need to break the analysis down into cases that cover all of the common characteristics, and label the cases so I can refer back to them easily. I am going to depend heavily upon the variables w_L and w_U to separate the cases into logical groups:

- w_L is the least value of w for which B(w) transitions from negative to positive values; and
- w_U is the least value of w for which B(w) transitions from positive to negative values;

where "L" stands for the lower bound on an interval, and "U" stands for upper bound on the interval. Then $\forall w \in \left[[w_L, w_U] \cap [0, \infty]\right]$: $B(w) \geq 0$. $w_L < w_U$. Then, $w_L \in [0, \infty)$ and $w_U \in (w_L, \infty]$.

I know I have not captured this quite right. Sometimes either w_L or w_U may not exist, or there may be multiple times when B(w) crosses the w axis. The above statement is true in most reasonable cases, but I go into some unreasonable cases where it is not exactly true.

5.1 Cases for w = 0

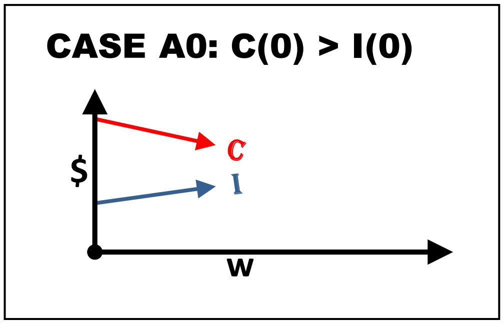
5.1.1 Case A: The Lower Extremity

I will start with what happens at w = 0, and work towards the right. I'll call this Case A, but it can be broken down into several sub-cases and sub-sub-cases. So, when w = 0 one of three conditions exist:

- **Sub-case A0** – costs exceed income, and benefits are less than zero. In which case w_L must exist, and must be > 0.
- **Sub-case A1** – costs equal income, and benefits are equal to zero, and w_L is coincident with w = 0, i.e. $w_L = 0$. This can be broken down further into two parts.

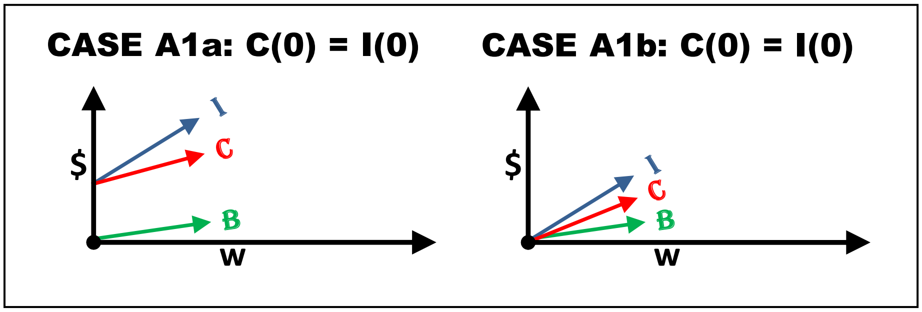
- **Sub-case A2** – income exceeds costs, and benefits are greater than zero. In this case w_L does not necessarily exist. This can be broken down further into two parts.

5.1.2 Case A0: C(0) > I(0)



Occurs outside of an interval where B(w)>0, but it implies, together with Cond 04, that for some $w_L > 0$ $C(w_L) = I(w_L)$. See Case B for discussion of that circumstance. See **Annex – Case A0** for a not-very-interesting example of this case.

5.1.3 Case A1: C(0) = I(0)



That is $w_L = 0$. This case has two sub-cases that I want to look at. C and I may equal each other and equal a positive value, or they may equal each other, and equal zero. I'll call these sub-cases B1 and B2 respectively. Since I know that, to the immediate right of w_L, B(w) > 0, therefore I(w) > C(w) to the right of w_L. So, the two sub-cases look as shown.

5.1.4 Sub-case A1a: I(0) > 0

I can say something about the value of each of the seven functions of interest:

- **Income**: I(0) > 0; defines the sub-case A1a.
- **Costs**: C(0) = I(0); defines the case A1.
- **Benefits**: B(0) = 0; defines the case A1.
- **Time**: T(0) = positive constant T; a simplifying assumption.
- **Power**: P(0) = B(0)/T(0) = 0/T(0) = 0; assuming that T(w) is well-behaved.
- **Efficiency**: η(0) = B(0)/I(0) = 0/I(0) = 0.
- **ROI**: R(0) = B(0)/C(0) = 0/C(0) = 0.

5.1.5 Sub-case A1b: I(0) = 0

Again, I can say something about the value of each of the six functions of interest:

- **Income**: I(0) = 0; defines the sub-case A1b.
- **Costs**: C(0) = I(0) = 0; defines the case A1.
- **Benefits**: B(0) = 0; defines the case A1.
- **Time**: T(0) = positive constant T; a simplifying assumption.
- **Power**: P(0) = B(0)/T(0) = 0/T(0) = 0; assuming that T(w) is well-behaved.
- **Efficiency**: η(0) = B(0)/I(0) = 0/0; an indeterminate value.
- **ROI**: R(0) = B(0)/C(0) = 0/C(0) = 0/0; an indeterminate value.

I must use limits to examine efficiency and ROI. Starting with efficiency:

$\lim_{w\to 0^+} \eta(w) = \frac{\lim_{w\to 0^+}[B(w)]}{\lim_{w\to 0^+} I(w)} = \frac{\lim_{w\to 0^+}[I(w) - C(w)]}{\lim_{w\to 0^+} I(w)}$	Equ 05

Simple substitution (Rule 07) results in the indeterminate value. So, let me try for a linear approximation of I(w) and C(w) near w=0. Let $\delta \in (0, 1]$ be a positive real number very close to zero. Then, since I(w) and C(w) are smooth functions of w, there is a δ small enough such that I(w) and C(w) can be reasonably (arbitrarily closely) represented on the interval [0, δ] by a linear function having intercepts at α_I and α_C respectively, and slopes β_I and β_C respectively. Assuming that B(w) is positive over this interval define functions I* and C* as:

$I(w) \cong I^*(w) \equiv \alpha_I + \beta_I w$	Equ 06
$C(w) \cong C^*(w) \equiv \alpha_C + \beta_C w$	Equ 07

Where $I^*(w)$ and $C^*(w)$ are the approximating linear functions defined on [0, δ]. I am using an asterisk notation to denote the linear approximations of smooth curves around the value w = 0.

Substituting these into equation 05 I get:

$\lim_{w\to 0^+} \eta(w) = \frac{\lim_{w\to 0^+}\left[\alpha_I + \beta_I w - \left(\alpha_C + \beta_C w\right)\right]}{\lim_{w\to 0^+}\left[\alpha_I + \beta_I w\right]}$	Equ 08
$\lim_{w\to 0^+} \eta(w) = \frac{\lim_{w\to 0^+}\left[(\alpha_I - \alpha_C) + \left(\beta_I - \beta_C\right)w\right]}{\lim_{w\to 0^+}\left[\alpha_I + \beta_I w\right]}$	Equ 09

L'Hôpital's rule says that, if the limits of both f and g are zero, then:

$\lim_{w\to 0^+} \frac{f(w)}{g(w)} = \lim_{w\to 0^+} \frac{f'(w)}{g'(w)}$	Equ 10

Replacing both the numerator and denominator of equation 09 with their derivatives gives me:

$\lim_{w\to 0^+} \eta(w) = \frac{\lim_{w\to 0^+}\left[\beta_I - \beta_C\right]}{\lim_{w\to 0^+}\left[\beta_I\right]} = \frac{\beta_I - \beta_C}{\beta_I}$	Equ 11

So the slopes of the income and cost curves at w = 0 determine the efficiency. I note that the efficiency can be negative, according to this calculation. Such negative efficiencies cannot happen here given the restrictions placed on the curves so far. For example, if $I^*(0) = 0$ and $C^*(0) = 0$, and if $I^*(w) > C^*(w)$ for all $w \in [0, \delta]$, then β_I must be $> \beta_C$, and the limit must be positive. This circumstance changes for the next case.

Next, looking at ROI:

$\lim_{w\to 0^+} R(w) = \frac{\lim_{w\to 0^+}[B(w)]}{\lim_{w\to 0^+} C(w)} = \frac{\lim_{w\to 0^+}[I(w) - C(w)]}{\lim_{w\to 0^+} C(w)}$	Equ 12

Substitution equations 06 and 07 in here gives me:

$\lim_{w\to 0^+} R(w) = \frac{\lim_{w\to 0^+}[(\alpha_I - \alpha_C) + (\beta_I - \beta_C)w]}{\lim_{w\to 0^+}[\alpha_C + \beta_C w]}$	Equ 13

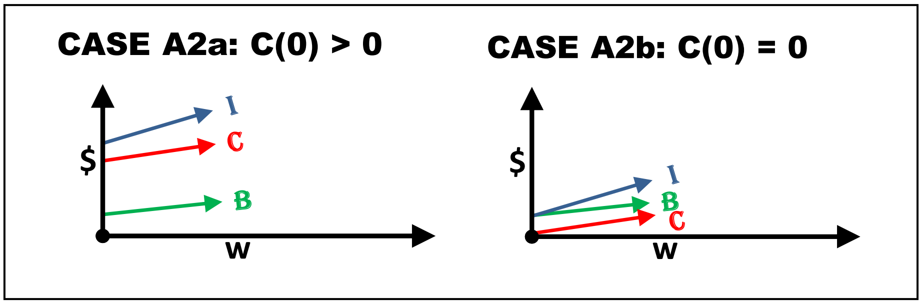
Using L'Hôpital's rule, again, to evaluate this I get:

$\lim_{w\to 0^+} R(w) = \frac{\lim_{w\to 0^+}[\beta_I - \beta_C]}{\lim_{w\to 0^+}[\beta_C]} = \frac{\beta_I - \beta_C}{\beta_C}$	Equ 14

Now, I know that, over the interval $[0, \delta]$, $C(w) < I(w)$ so β_C must be less than β_I. In fact the slope of the cost curve must be in the range $0 < \beta_C < \beta_I$. This implies that $\lim_{w\to 0^+} \eta(w) \in (0,1)$ and $\lim_{w\to 0^+} R(w) \in (0,\infty)$. Equations 11 and 14 are, I think, interesting results.

5.1.6 Case A2: C(0) < I(0)

For Case A, w = 0. I am now looking at the instance wherein, if the costs equal the income, it occurs somewhere to the left of zero and there are positive benefits at the point w = 0. Again, I want to break this into two sub-cases for when C(0) > 0 and when C(0) = 0 – sub-cases A2a and A2b respectively.



5.1.7 Sub-case A2a: C(0) > 0

I can say something about the value of each of the seven functions of interest:

- **Income**: I(0) > B(0) and C(0); defines the A2 case and the A2a sub-case.
- **Costs**: 0 < C(0) < I(0); defines the A2 case and the A2a sub-case.
- **Benefits**: I(0) > B(0) > 0; defines the A2 case.
- **Time**: T(0) = positive constant T; a simplifying assumption.
- **Power**: P(0) = B(0)/T(0) = a positive number; assuming that T(w) is well-behaved.
- **Efficiency**: η(0) = B(0)/I(0) = a positive number.
- **ROI**: R(0) = B(0)/C(0) = a positive number.

5.1.8 Sub-case A2b: C(0) = 0

I can say something about the value of each of the seven functions of interest:

- **Income**: I(0) = B(0) > 0; defines the A2 case and the A2b sub-case.
- **Costs**: C(0) = 0; defines the A2b sub-case.
- **Benefits**: B(0) = I(0) > 0; defines the A2b sub-case.
- **Time**: T(0) = positive constant T; a simplifying assumption.
- **Power**: P(0) = B(0)/T(0) = a positive number; assuming that T(w) is well-behaved.
- **Efficiency**: η(0) = B(0)/I(0) = 1.
- **ROI**: R(0) = B(0)/C(0) = B(0)/0 = ∞.

Again, I can use limits to examine η(0) a little more closely. Equation 05 applies here, also. Although simple substitution (Rule 07) no longer results in the indeterminate value, it still provides an intractable formula that gives me little insight. So, perhaps linear approximations will help me again. So, let me again use the linear approximations I^* and C^* near w=0 that were shown in Equations 05 through 09. I cannot use L'Hôpital's rule this time because I don't have an indeterminate limit.

I repeat equation 09 here as equation 15, for reference:

$$\lim_{w\to 0^+} \eta(w) = \frac{\lim_{w\to 0^+}\left[(\alpha_I - \alpha_C) + (\beta_I - \beta_C)w\right]}{\lim_{w\to 0^+}\left[\alpha_I + \beta_I w\right]} = \frac{\alpha_I - \alpha_C}{\alpha_I}$$	Equ 15

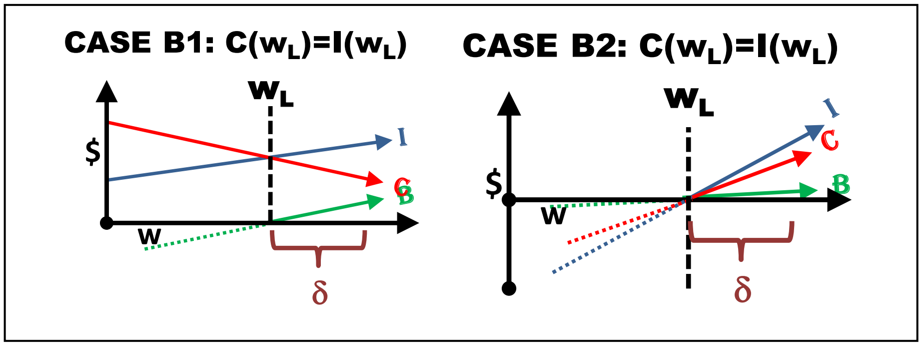
Substituting zero for w this time we get a ratio of intercepts. This formula works for both sub-cases A2a and A2b.

5.2 The Cases for $w_L > 0$

I have now completed, I think, the analysis for the extremity when w = 0. Now, I need to look at cases for w in the range 0 < w < ∞. By condition 5 there must be some w_L, some least value of w, for which $B(w_L) > 0$. I.e. there must be some least value of w for which there is a benefit achieved. This is true for cases B and C when w = 0, so, in those cases $w_L = 0$. In Case A0, however, w_L must be > 0, due to condition 05. So, at this point I need to return indirectly to Case A0 again and develop it further as Case B.

5.2.1 Case B: $I(w_L) = C(w_L)$, for $w_L > 0$

w_L is the least value of w for which there are benefits ≥ 0. This case is closely related to Case A0, due to Cond 04, in that w_L MUST exist, and whenever Case A0 is extant, Case B must also be extant.



This case is somewhat similar to Case A – differing only by reason of the fact that w is now non-zero. So, I expect much of the analysis from Case A also applies here.

More formally, there is some least positive value of the parameter w, denoted by w_L, and there is some positive number δ, such that the following three conditions are true:

- $\forall w \in [w_L-\delta, w_L)$: $I(w) < C(w)$
- $I(w_L) = C(w_L)$
- $\forall w \in (w_L, w_L+\delta]$: $I(w) > C(w)$

So, I am looking for the value of w where the income first equals then exceeds the costs, and benefits start to flow. Again, for the same reasons as before, I want to break this into two sub-cases.

5.2.2 Sub-case B1: $I(w_L) > 0$

In sub-case B1, interpretation at the point w_L seems fairly straight-forward.

- **Income**: $I(w_L) = C(w_L) > 0$; defines the B case and the B1 sub-case.
- **Costs**: $C(w_L) = I(w_L) > 0$; defines the B case and the B1 sub-case.
- **Benefits**: $B(w_L) = 0$; defines the B case.
- **Time**: $T(w_L)$ = positive constant T; a simplifying assumption.
- **Power**: $P(w_L) = B(w_L)/T(w_L) = 0$; assuming that T(w) is well-behaved.
- **Efficiency**: $\eta(w_L) = B(w_L)/I(w_L) = 0/I(w_L) = 0$.
- **ROI**: $R(w_L) = B(w_L)/C(w_L) = 0/C(w_L) = 0$.

See **Annex: Case B1** for an example.

5.2.3 Sub-case B2: $I(w_L) = 0$

In sub-case B2, interpretation at the point w_L is more tricky again.

- **Income**: $I(w_L) = C(w_L) = 0$; defines the B case and the B2 sub-case.
- **Costs**: $C(w_L) = I(w_L) = 0$; defines the B case and the B2 sub-case.
- **Benefits**: $B(w_L) = 0$; defines the B case.
- **Time**: $T(w_L)$ = positive constant T; a simplifying assumption.
- **Power**: $P(w_L) = B(w_L)/T(w_L) = 0/T(w_L) = 0$; assuming that T(w) is well-behaved.
- **Efficiency**: $\eta(w_L) = B(w_L)/I(w_L) = 0/0$; an indeterminate.
- **ROI**: $R(w_L) = B(w_L)/C(w_L) = 0/0$; an indeterminate.

So sub-case B2 results in an indeterminate $\eta = 0 / 0$ as before. I think, therefore, that equations 05-11 also apply here, and L'Hôpital's rule can be used to evaluate the limit again.

I copy Equ 11 here and renumber it as equation 16, for reference, changing the point at which the limit is evaluated from w=0 to $w=w_L$:

$$\lim_{w \to w_L^+} \eta(w) = \frac{\lim_{w \to w_L^+}[\beta_I - \beta_C]}{\lim_{w \to w_L^+}[\beta_I]} = \frac{\beta_I - \beta_C}{\beta_I}$$	Equ 16

where β_I and β_C are the slopes of the lines $I^*(w)$ and $C^*(w)$ that approximate I(w) and C(w) over the delta interval $[w_L, w_L+\delta]$.

The efficiency then is again a ratio of slopes. It would appear that it is a positive value, being zero only if the costs are rising at an equal pace to the income. But, if the costs are not rising at all over that delta interval, then the efficiency can be close to, or equal to, one.

My conclusion then seems to be that $0 \le \eta(w_L) \le 1$ for sub-case B2. This defies my intuition. Case B2 has point values for $I(w_L)$ and $C(w_L)$ and is merely an extremity, a extreme version, of Case B1. Why is it so dramatically different?

It seems that, as $I(w_L)$ approaches zero the lower part of the loop becomes more and more closely asymptotic to the x axis, and the corner on the loop becomes closer and closer to a right angled turn, that when $I(w_L)$ is in fact equal to zero, the lower part of the loop has disappeared into the x axis, and the corner has become right angled. It is no longer a loop, but a bona fide concave-downwards (CCD) Goldilocks curve. Curiously, the right-most corner of such a curve need not be at the point $(P, \eta) = (0, 1)$, but it seems η can have any value at all between 0 and 1. See **Annex – Case B2** for an example.

By similar logic, the return on investment, when expressed as a function of w [e.g. R(w)], can be show to be:

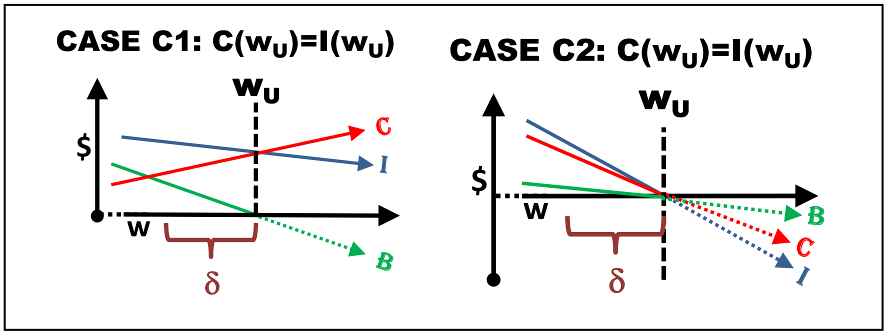
$$\lim_{w \to w_U^-} R(w) = \frac{\lim_{w \to w_U^-}[\beta_I - \beta_C]}{\lim_{w \to w_U^-}[\beta_C]} = \frac{\beta_I - \beta_C}{\beta_C}$$	Equ 17

5.3 Cases for w_U exists, and $w_U < \infty$

That is, these are the cases for which w_U is a finite number, and not infinite. w_U is the upper bound on the range of w that provides non-zero benefits. I would expect that these cases would be similar in structure to Case B.

5.3.1 Case C: $I(w_U) = C(w_U)$, for $w_U < \infty$

This case is closely related to Case B, and is in some ways the flip side of Case B.



More formally, for the parameter w there is some least positive value, denoted by w_U, and there is some positive number δ, such that the following three conditions are true:

- $\forall w \in [w_U - \delta, w_U]$: $I(w) > C(w)$
- $I(w_U) = C(w_U)$
- $\forall w \in (w_U, \infty]$: $I(w) < C(w)$

So, I am looking for the value beyond which the income no longer exceeds the costs, and benefits stop flowing. Again, for the same reasons as before, I want to break this into two sub-cases.

5.3.2 Sub-case C1: $I(w_U) > 0$

In sub-case C1, interpretation at the point w_U seems fairly straight-forward.

- **Income**: $I(w_U) = C(w_U) > 0$; defines the C case and the C1 sub-case.
- **Costs**: $C(w_U) = I(w_U) > 0$; defines the C case and the C1 sub-case.
- **Benefits**: $B(w_U) = 0$; defines the C case.
- **Time**: $T(w_U)$ = positive constant T; a simplifying assumption.
- **Power**: $P(w_U) = B(w_U)/T(w_U) = 0/T(w_U) = 0$; assuming that T(w) is well-behaved.
- **Efficiency**: $\eta(w_U) = B(w_U)/I(w_U) = 0/I(w_U) = 0$.
- **ROI**: $R(w_U) = B(w_U)/C(w_U) = 0/C(w_U) = 0$.

See **Annex: Case C1** for an example.

5.3.3 Sub-case C2: $I(w_U) = 0$

In sub-case B2, interpretation at the point w_U is more tricky again.

- **Income**: $I(w_U) = C(w_U) = 0$; defines the C case and the C2 sub-case.
- **Costs**: $C(w_U) = I(w_U) = 0$; defines the C case and the C2 sub-case.
- **Benefits**: $B(w_U) = 0$; defines the C case.
- **Time**: $T(w_U)$ = positive constant T; a simplifying assumption.
- **Power**: $P(w_U) = B(w_U)/T(w_U) = 0/T(w_U) = 0$; assuming that T(w) is well-behaved.
- **Efficiency**: $\eta(w_U) = B(w_U)/I(w_U) = 0/0$; an indeterminate.
- **ROI**: $R(w_U) = B(w_U)/C(w_U) = 0/0$; an indeterminate.

As I thought, this is extremely similar to Case B. So sub-case C2 results in an indeterminate $\eta = 0 / 0$ as before. Equations 05-11 also apply here, and L'Hôpital's rule can be used to evaluate the indeterminate limit again.

I copy Equ 11 here and renumber it as equation 18, for reference, changing the point at which the limit is reached from w=0 to w=w_U, and evaluating the limit from the left, instead of from the right.

$$\lim_{w \to w_U^-} \eta(w) = \frac{\lim_{w \to w_U^-} [\beta_I - \beta_C]}{\lim_{w \to w_U^-} [\beta_I]} = \frac{\beta_I - \beta_C}{\beta_I}$$	Equ 18

where β_I and β_C are the slopes of the lines $I^*(w)$ and $C^*(w)$ that approximate I(w) and C(w) over the delta interval [w_L-δ, w_L].

The efficiency then is again a ratio of slopes. It would appear that it is a positive value, being zero only if the costs are falling at an equal pace to the falling income. But, if the costs are not falling at all over that delta interval, then the efficiency can be close to, or equal to, one. My conclusion then seems to be that $0 \leq \eta(w_U) \leq 1$ for sub-case C2. See **Annex – Case C2** for an example.

By similar logic, the return on investment, when expressed as a function of w [e.g. R(w)], can be showN to be:

$$\lim_{w \to w_U^-} R(w) = \frac{\lim_{w \to w_U^-} [\beta_I - \beta_C]}{\lim_{w \to w_U^-} [\beta_C]} = \frac{\beta_I - \beta_C}{\beta_C}$$	Equ 19

This is dramatically different from efficiency. If the slope β_C is close to β_I then the $R(w_U)$ is close to zero. However, if the slope β_C is close to zero, then $R(w_U)$ is close to infinite. So $0 \leq R(w_U) \leq \infty$. That is peculiar, indeed. I cannot think of a real-world economic circumstance that would lead to an infinite ROI like this, but the mathematics is intriguing.

5.4 Cases involving w→∞

This where it gets really tricky. The tricky part, I think, is not so much the mathematics as the laying out of cases that are very general, but that might have some reasonable meaning. I somewhat arbitrarily set boundary conditions 01 and 02 so that I(w) and C(w) have upper bounds on their values for all w < infinity, and together they imply condition 03 – the restrictions on B(w). By choosing those conditions, I may have excluded some interesting cases from discussion here, but I think they are reasonable conditions.

Nevertheless, those three conditions, being quite general, still allow for all kinds of behaviour, of which Case C is merely one obvious example, in which the value of B(w) eventually goes negative at some finite value of w. Now, it might go positive again and negative again many times, but Case C only looks at the least value of w for which it goes negative, and that is fine. I cannot imagine any use in resolving crazy cases for which B(w) dips above and below zero two or more times as w increases, but I have done some experimentation with that. For example, I could consider that for each interval of the domain of w over which B(w) crosses zero, rises to some height, falls back, and sinks below zero – for each such interval it can be considered an interval that is described by Case B at the lower end and case C at the upper end. I lump such this repetitive behaviour under Case D.

5.4.1 Case D: Repeating intervals of Positive B(w)

I played with this for a while and came up with one curve set that I found interesting. I decided to make the income curve an attenuated sine curve with a formula like this: $I = a\,sin(bw - c) + d$. Similarly, the cost curve has a formula like this: $I = a\,sin(bw - c) + d$. This allows me to make the two curves interact repeatedly within the domain of my w parameter [0, 99]. And I can make them interact in different ways. See **Annex – Case D** for an example.

5.4.2 Case E: ∀w∈[δ, ∞]: B(w)>0

So, the only circumstances not yet covered are those for which B(w) rises above zero and stays above zero in some δ-neighbourhood of ∞. But it is still possible, within the bounds of conditions 01 and 02, for the B(w) curve to approach a limiting value, or to wander back and forth between zero and I_{max} in some sort of regular or chaotic fashion as w approaches infinity. There is still a huge amount of room for varied behaviour.

I guess, at this point, I want to focus on the cases that stabilize, and exclude those that wander between the upper and lower bounds. So, let me see if I can make my ideas a little more formal and find a way to classify the wide range of possible behaviours. Once w is large enough that B(w) is positive for some δ-neighbourhood of ∞, there are four main classes of behaviour. Denote $\lim_{w\to\infty} I(w)$ as I_∞. Denote $\lim_{w\to\infty} C(w)$ as C_∞. Each of these may exist, or may not exist. So, Case E has the following sub-cases:

- **Sub-case E1**: I_∞ exists, and C_∞ exists.
- **Sub-case E2**: I_∞ exists, and C_∞ does not exist.
- **Sub-case E3**: I_∞ does not exist, and C_∞ exists.
- **Sub-case E4**: I_∞ does not exist, and C_∞ does not exist.

5.4.3 Sub-case E1: Both I_∞ and C_∞ exist

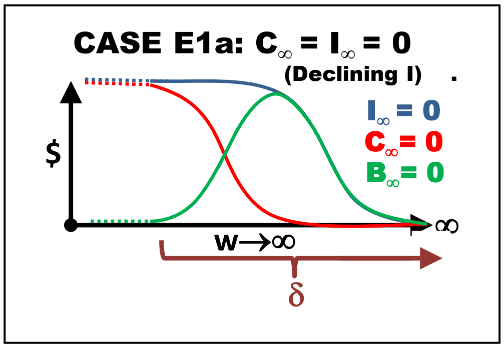
If both limits exist, then B_∞ also exists and $B_\infty = I_\infty - C_\infty$. In sub-case E1, interpretation as w→∞ is relatively straightforward:

- **Income**: $I(w\to\infty) = I_\infty$; a constant; defines the E case and the E1 sub-case.
- **Costs**: $C(w\to\infty) = C_\infty$; a constant in the interval $[0, I_\infty]$; defines the E case and the E1 sub-case.
- **Benefits**: $B(w\to\infty) = I_\infty - C_\infty = B_\infty$; a constant in the interval $[0, I_\infty]$. I note that B_∞ can be zero while B(w) is positive on the δ-neighbourhood of infinity.
- **Time**: $T(w\to\infty)$ = positive constant T; a simplifying assumption.
- **Power**: $P(w\to\infty) = B_\infty/T(w\to\infty)$; assuming that T(w) is well-behaved over the interval w∈[δ, ∞].
- **Efficiency**: $\eta(w\to\infty) = B_\infty/I_\infty$; a positive value in the interval (0, 1], or an indeterminate.
- **ROI**: $R(w\to\infty) = B_\infty/C_\infty$; a positive value in the interval (0, ∞], or an indeterminate.

With respect to power, to this point, I have been intentionally dodging the issue of how I should characterize a "well-behaved" T(w) by simply assuming it is constant. More generally, consistent with the uses I have made of it so far, I have assumed that it is a bounded function of w such that $T(w)\in[T_{min}, T_{max}]$ where $T_{min} \neq 0$ and T_{max} is finite. I think that exploring the role of time is a topic for a different NTF as this one is already very long. Let me just make a few pertinent observations here:

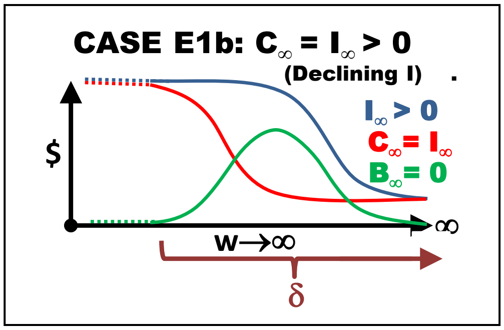
- The role of T(w) is extremely relevant to the shape of power-efficiency (or Goldilocks) curves as it is a factor of power.
- T(w) might be well-behaved (e.g. have finite upper and lower bounds) on some interval $w \in [w_L, w_U]$ of finite length, and that is not an unreasonable assumption, and is consistent with my analysis for Cases A, B, C and D. However an assumption that T(w) is well-behaved in that way on the interval $w \in [\delta, \infty]$ which has infinite length is far more tenuous.
- But, perhaps more to the point, in my analysis of the AM I found that time can be expressed as a function of efficiency (i.e. η) having the form $T(\eta) = C \times (\eta + 1)/(\eta - 1)$. Then efficiency approaching 1 causes time to soar towards ∞ and power to crash towards an indeterminate value of 0/∞ in many cases. Such behaviour will alter the shape of all of the Goldilocks curves I have examined so far.
- This is only slightly worrisome, as I think it will simply cause the Goldilocks curves to droop as $\eta \rightarrow 1$, moving the point of maximum power to the left, but leaving the central thesis (that all such curves show maximum power at some intermediate level of efficiency) intact.
- BUT, for now, I will continue on assuming that T(w) = finite positive constant is a workable characterization of the relation between time and w. For this kind of analysis, I don't need to model time exactly; I only need to know the restrictions on its general behaviour in various limiting situations.

With respect to efficiency and ROI, there can be at least three sub-sub-cases:



- **Sub-sub-case E1a**: $I_\infty=0$, $B_\infty=0$, which implies that $C_\infty=I_\infty$. In addition, I(w) approaches the asymptote from above.
  - **Power**: $P(w\rightarrow\infty) = B_\infty/T(w\rightarrow\infty) = 0/T(w\rightarrow\infty) = 0$; assuming that T(w) is well-behaved over the interval $w \in [\delta, \infty]$.
  - **Efficiency**: $\eta(w\rightarrow\infty) = B_\infty/I_\infty = 0/0$ = indeterminate.
  - **ROI**: $R(w\rightarrow\infty) = B_\infty/C_\infty = 0/0$ = indeterminate.

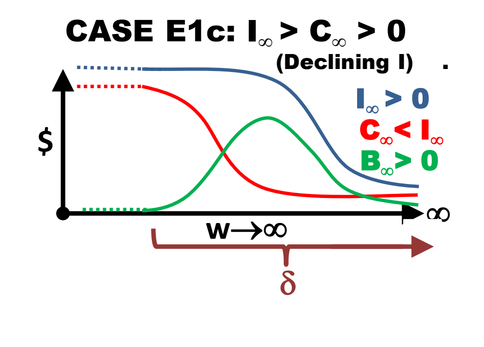
The I, C and B curves are all asymptotic to zero as $w\rightarrow\infty$, so I don't think I can make a linear approximation work (all of the slopes are zero) to evaluate limits. I suppose I could use quadratic, trinomial, or higher orders of approximation, and apply LHôpital's rule multiple times to examples to test whether it would work, but I am not sure what that would prove. It is easy to think of examples of curve sets (e.g. exponentials for I and C) for which LHôpital's rule cannot provide an answer. So, I don't know how one might analytically evaluate the indeterminate values for efficiency and for ROI, in this case. I suppose the pragmatic answer is to graph them and estimate the value of the limit. Perusal of the graphs at **Annex E1a** gives me some idea of the values. $\eta(w\rightarrow\infty) = 1$. $R(w\rightarrow\infty) = 0$.



- **Sub-sub-case E1b**: $I_\infty > 0$, $C_\infty = I_\infty$ and $B_\infty = 0$. I_∞ necessarily approaches the asymptote from above.

- **Power**: $P(w\to\infty) = B_\infty/T(w\to\infty) = 0/T(w\to\infty) = 0$; assuming that T(w) is well-behaved over the interval $w\in[\delta, \infty]$.
- **Efficiency**: $\eta(w\to\infty) = B_\infty/I_\infty = 0/I_\infty = 0$.
- **ROI**: $R(w\to\infty) = B_\infty/C_\infty = 0/C_\infty = 0$.

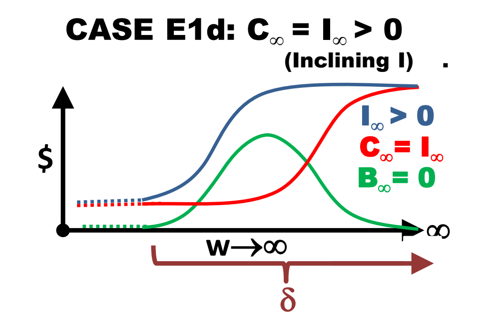
There is no ambiguity about the values for efficiency and ROI.

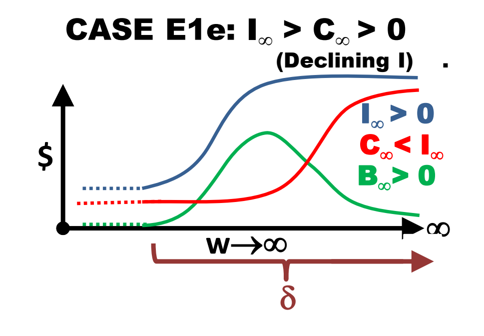


- **Sub-sub-case E1c**: $I_\infty > 0$, $C_\infty < I_\infty$ and $B_\infty > 0$. I_∞ necessarily approaches its asymptote from above.
  - **Power**: $P(w\to\infty) = B_\infty/T(w\to\infty)$ = a positive number; assuming that T(w) is well-behaved over the interval $w\in[\delta, \infty]$.
  - **Efficiency**: $\eta(w\to\infty) = B_\infty/I_\infty$ = a positive number.
  - **ROI**: $R(w\to\infty) = B_\infty/C_\infty$ = a positive number.

  Again, as in case E1c, there is no ambiguity about the values for efficiency and ROI.

- **Sub-sub-cases E1d and E1e**: $I_\infty \in (0, I_{max}]$. I.e. $I_\infty > 0$, as in case E1a. In addition, I(w) approaches the asymptote from below.
  - **Costs**: $C(w\to\infty) = C_\infty$; $0 \le C_\infty \le I_\infty$; a constant in the interval $[0, I_\infty]$; defines the E1 sub-case and the E1d and E1e sub-sub-cases.
  - **Benefits**: $B(w\to\infty) = I_\infty - C_\infty = B_\infty$; a constant in the interval $[0, I_\infty]$.
  - **Time**: $T(w\to\infty)$ = positive constant T; a simplifying assumption.
  - **Power**: $P(w\to\infty) = B_\infty/T(w\to\infty)$; zero or a positive number; assuming that T(w) is well-behaved over the interval $w\in[\delta, \infty]$.
  - **Efficiency**: $\eta(w\to\infty) = B_\infty/I_\infty$; a value in the interval [0, 1].
  - **ROI**: $R(w\to\infty) = B_\infty/C_\infty = (I_\infty - C_\infty)/C_\infty$; a positive value in the interval $[0, \infty]$.





See **Annexes E1a, E1b and E1d** for examples of these cases.

5.4.4 Sub-case E2: I_∞ exists but C_∞ does not

In sub-case E2, interpretation as $w\to\infty$ is as follows:

- **Income**: $I(w\to\infty) = I_\infty$; a constant in the interval $[0, I_{max}]$; defines the E case and the E2 sub-case.
- **Costs**: $C(w\to\infty)$ varies on the interval $[0, I_\infty)$; defines the E case and the E2 sub-case. C(w) cannot equal I_∞, or the curve set is in breach of the Case E requirement that $B(w) > 0$ in the δ-neighbourhood of ∞.
- **Benefits**: $B(w\to\infty) = I_\infty - C(w\to\infty)$ varies on the interval $(0, I_\infty]$.

- **Time**: $T(w\to\infty)$ = positive constant T; a simplifying assumption.
- **Power**: $P(w\to\infty) = B(w\to\infty)/T(w\to\infty)$; assuming that T(w) is well-behaved over the interval $w\in[\delta, \infty]$. $P(w\to\infty)$ varies on an interval $(0, P_{max}]$.
- **Efficiency**: $\eta(w\to\infty) = B(w\to\infty)/I_\infty$ varies in the interval (0, 1].
- **ROI**: $R(w\to\infty) = B(w\to\infty)/ C(w\to\infty) = [I_\infty - C(w\to\infty)]/ C(w\to\infty)$ varies in the interval $(0, \infty]$.

That is, costs, benefits, power, efficiency and ROI all vary continuously as $w\to\infty$. This is highly unlikely as a real-world phenomenon, and not of any real interest, but does produce some fantastic scatter graphs, not entirely dis-similar to those of Case D.

I have not provided an annex with an example for this case.

5.4.5 Sub-case E3: I_∞ does not exist, but C_∞ does

To think this through, I need a little specialized notation. The δ-neighbourhood of ∞ is written as $[\delta, \infty]$ and is a sub-set of the domain of the variable w. I(w) must be between 0, at the very least, and Imax, at the very greatest. So, within that δ-neighbourhood of ∞, I(w) must have a minmum value of $I_{min, \delta}$ and a maximum value of $I_{max, \delta}$ where the interval $[I_{min, \delta}, I_{max, \delta}] \subset [0, I_{max}]$. It is possible that the maximum value of I_{max} was attained outside of $[\delta, \infty]$, so I_{max} may be greater than $I_{max, \delta}$. I can simplify $[I_{min, \delta}, I_{max, \delta}]$ by writing it as $[I_{min}, I_{max}]_\delta$, meaning the lower and upper bounds of the interval are determined by checking the variables within the δ-neighbourhood of ∞.

In sub-case E3, interpretation as $w\to\infty$ is as follows:

- **Income**: $I(w\to\infty)$ varies on the interval $[I_{min}, I_{max}]_\delta$; defining Case E3.
- **Costs**: $C(w\to\infty)$ varies on the interval $[0, I_{max})_\delta$; defines the E case and the E3 sub-case. C(w) cannot equal $I_{max, \delta}$, or the curve set is in breach of the Case E requirement that $B(w) > 0$ in the δ-neighbourhood of ∞.
- **Benefits**: $B(w\to\infty) = I(w\to\infty) - C(w\to\infty)$ varies on the interval $(0, I_{max, \delta}]$.
- **Time**: $T(w\to\infty)$ = positive constant T; a simplifying assumption.
- **Power**: $P(w\to\infty) = B(w\to\infty)/T(w\to\infty)$; assuming that T(w) is well-behaved over the interval $w\in[\delta, \infty]$. $P(w\to\infty)$ varies on an interval $(0, P_{max}]$.
- **Efficiency**: $\eta(w\to\infty) = B(w\to\infty)/ I(w\to\infty)$ varies in the interval (0, 1].
- **ROI**: $R(w\to\infty) = B(w\to\infty)/ C(w\to\infty) = [I_\infty - C(w\to\infty)]/ C(w\to\infty)$ varies in the interval $(0, \infty]$.

In this case, all six of the functions that vary with w continue to vary everywhere within the δ-neighbourhood of ∞.

I have not provided an annex with an example for this case.

5.4.6 Sub-case E4: Neither I_∞ nor C_∞ exist

Intuitively, this will be similar to Case E3, except there might also be a set of lower and upper bounds on the variance of C(w). I think it is not worth exploring, as case E3 was pretty much a bust.

I have not provided an annex with an example for this case.

6 Summary

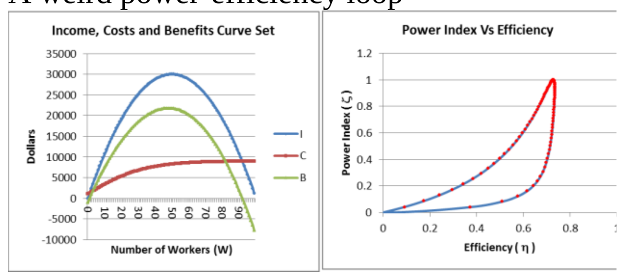
Wow! Here I am on page 19 of what I thought would be a short NTF, with another 20+ pages of graphs. What can I say in summary? I suppose I should do "lessons learned" and "yet to do" lists.

6.1 Lessons Learned

Here is a bulleted list of things that I did not know before I started the series of partial and faulty exercises that culminated in this NTF:

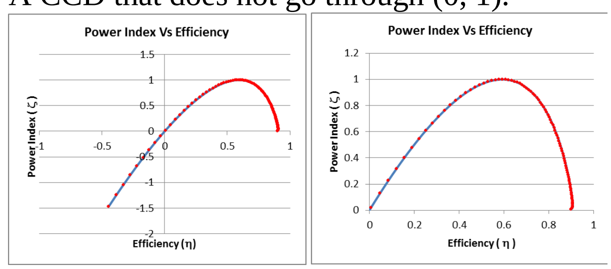
- Not all of the power-efficiency curves coming out of this exercise can be converted to concave-downwards (CCD) unit maps, as I previously thought. In fact, it is rather difficult to get a power-efficiency curve to go through both (P, η) = (0, 0) and (P, η) = (0, 1). In this exercise, most power-efficiency curves are not CCD curves, but are loops. Loops are not Cartesian functions, and cannot be expressed as P(η) using Cartesian coordinates, but, rather, are relations that can only be easily graphed as scatter graphs. Perhaps, they can be captured analytically using polar equations and polar coordinates, but I think that even those would have some problems. For example, the figure to the right here shows one power-efficiency graph that is not a function in either Cartesian or polar coordinates.

A weird power-efficiency loop



- All of the power-efficiency graphs that I have been able to produce so far, in this exercise, or in any other, are either CCD "hump-backed" curves, or loops. But, in every case, the curve has the property that the maximum power occurs when efficiency is at some intermediate value of neither 0 nor 1. So I have started referring to them as Goldilocks curves, being not too hot, not too cold, but just right.

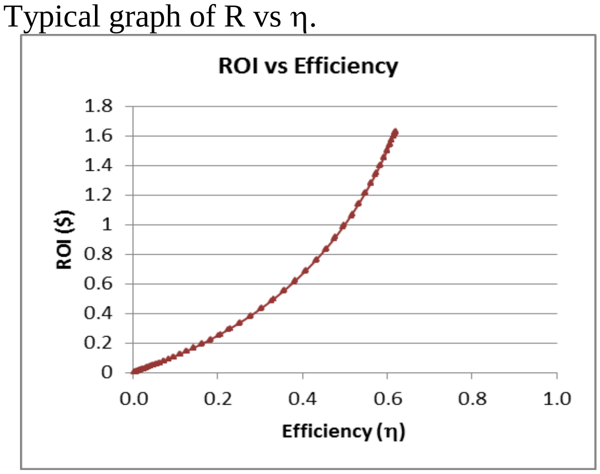
A CCD that does not go through (0, 1).



See Annex – Case A1b

- Cases B and C are the most insightful, with strange things happening at the extremities of an interval denoted as w∈[w_L, w_U]. It is determined by the conditions B(w_L) = 0; B(w_U) = 0; and $B \forall w \in (w_L, w_U)$: $B(w) > 0$. At the critical end points the benefits are zero, and the power of the benefits is also

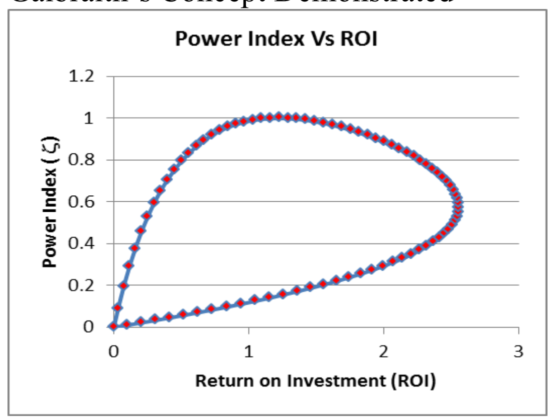
Typical graph of R vs η.



therefore zero at those end points. But, two sorts of things happen. If the income and costs are greater than zero at these critical end points, then a Goldilocks loop occurs. But, if the income and costs are also zero at the end points, then the efficiency (η(w)) and return on investment (ROI(w), or just R(w)) are expressed as ratios of slopes: $\eta(w) = (\beta_I - \beta_C)/\beta_I$; and $R(w) = (\beta_I - \beta_C)/\beta_C$. This strange result allows for the end points of the loop to be anchored somewhere else on the interval [0, 1] other than at the origin where (P,η)=(0,0). It also allows for the ROI to be very large.

- ROI and efficiency always peak for the same value of the controlling parameter w. And points for R vs η always seem to fall on the same line. A little math shows me that the definitions are not independent, and that, in fact, I can come up with an analytic expression for R in terms of η: $R = \eta/(1 - \eta)$.
- However, power and ROI do not peak at the same value. This leads directly to the ideas of John Kenneth Galbraith who argued (1967, 1972, and described by Mercadier at Ref K) that large modern firms no longer try to maximize ROI, but, rather, try to maximize their size and their probability of persistence. They feed returns back into mergers, acquisitions, growth, expansion, research, and resilience, diverting some profits from returns to the investor. This is an anomaly of economic theory and behaviour that I always thought existed, but did not understand before. I now see this as an effect of the MPP working in persistent modern large firms.

Galbraith's Concept Demonstrated



See Case B1.

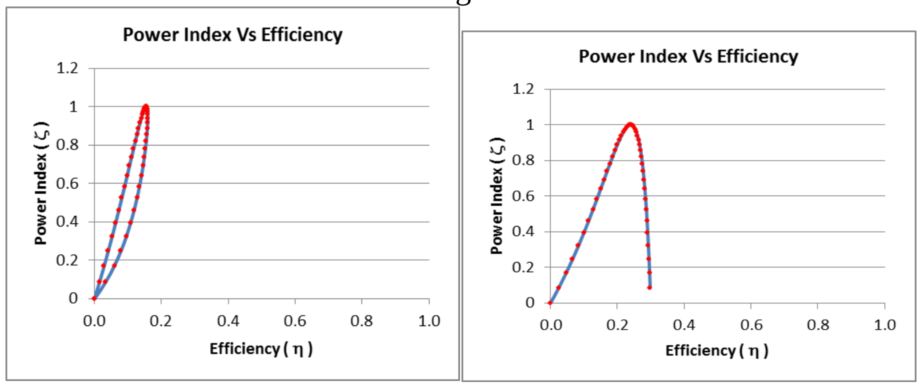
- I really need to figure out some way to allow time to vary, and see what the implications are. That will probably be another (hopefully shorter) note. I have described the way I have handled time, and the consequences of that assumption in this NTF, in three places: 4.1.2, 4.1.5, and 5.4.3.

6.1.1 Diminishing Returns

The whole purpose of this exercise, starting with the spreadsheet and NTF at Refs I and J, was to understand the connection between H. T. Odum's MPP and the phenomenon of diminishing returns.

Here, I need to remind myself that I am NOT dealing with diminishing returns as time proceeds. These results are all static, looking at w as a variable unrelated to time. I am talking about diminishing returns associated with some parameter of a business model such as number of workers.

Sub-cases C1 and C2 – Diminishing Returns

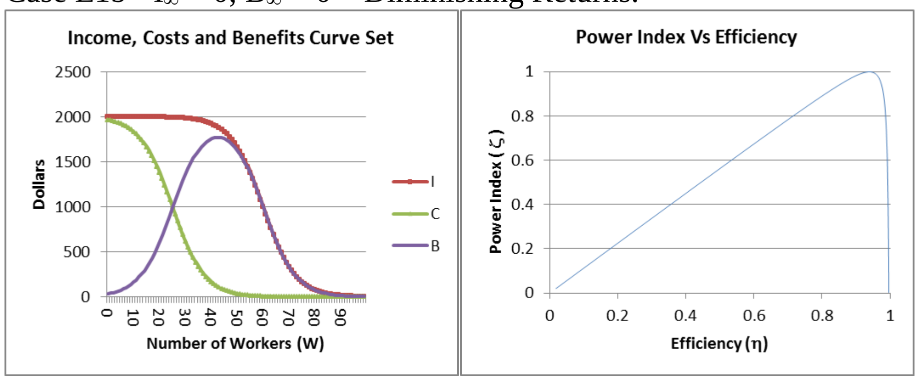


Goldilocks Loop Goldilocks Hump

So, here is a summary of what I think I have learned wrt that phenomenon:

- Case C is entirely a study of diminishing returns. For values of w less than w_L B(w) is greater than zero, but smoothly dropping to zero at w_L. This case is divided into two sub-cases.
- Sub-case C1 addresses those curve sets for which $I(w_L) > 0$. This results in a Goldilocks loop, of sorts, as shown here to the right.
- Sub-case C2 addresses those curve sets for which $I(w_L)=0$. This results in a Goldilocks CCD, of sorts, as shown below the other. This is approaching the type of hump-backed curve I was (naively) expecting to find.

Case E1b - $I_\infty = 0$, $B_\infty = 0$ – Diminishing Returns.

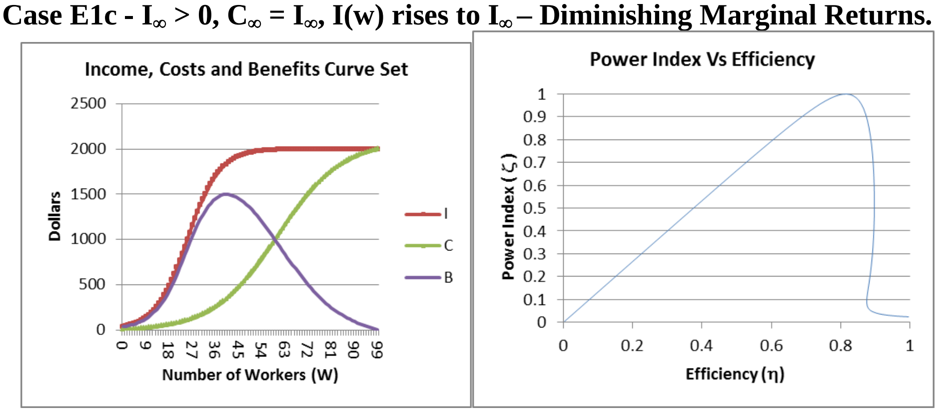


Goldilocks CCD Hump

- Sub-cases E1a, E1b and E1c are other cases that demonstrates diminishing returns, where $\lim_{w\to\infty} B(w) = 0$. In fact, every case for which B(w) peaks and then diminishes is a case of diminishing returns. Power peaks when B(w) peaks (assuming T(w) is well-bahaved).

6.1.2 Diminishing Marginal Returns

Case E1c - $I_\infty > 0$, $C_\infty = I_\infty$, I(w) rises to I_∞ – Diminishing Marginal Returns.



Goldilocks CCD Loopy Hump

When benefits continue to rise as $w\to\infty$, but are asymptotic to some value B_∞, then the returns are certainly not diminishing, but the rate of rise of the returns is diminishing. This circumstance is called diminishing marginal returns. You can think of this as a declining slope on the benefits curve. So, the returns, per unit of investment, start to diminish as the point when dB(w)/dw = zero (at the peak, if there is one) and starts to turn negative.

In the classic version, adding one more unit of cost (w = workers) “at the margin” causes an increase in profits, but the size of the increase diminishes, until the incremental increase in cost is larger than the incremental increase in profits. This would be included in Case C, since, at some point, costs equal income and profits plummet to zero. But whether extreme or classic, my goal in this exercise is to understand the connection to the MPP. Case E1c is, I think, an extremely interesting case of diminishing marginal returns, as the Goldilocks curve has both both loop and hump characteristics, being a hybrid of the two. Of course, power (using Odum’s definition, as applied by analogy to financial data) is at a maximum when the benefits curve (profit curve) is at a maximum, and this corresponds to the configuration of parameters for which marginal returns are maximized. And, this seems to be in agreement with Galbraith’s opinions about large modern corporations.

6.2 Yet-To-Do

Figure how “nature’s time regulator” fits into this.

7 Annexes

These annexes to the main NTF each contain material from the Ref L spreadsheet in which I experimented with various combinations of curves. In each annex there are, at least, two kinds of figures: Control Panels, and Graphs.

Control Panels – These figures are screen grabs from the spreadsheet showing the parameters for the particular curves used, and the first few data elements from each generated data series. Within each control panel there are the following elements:

- A generic parameterized formula for each of income [I(w)] and costs [C(w)]. These formulae have small-letter variable names for the parameters, such as a, b, c, d, f, g. I avoid using e, i and l for such parameters.
- A parameter table for each of I(w) and C(w), allowing me to alter the shape and location of each curve as I will. The yellowed cells are where values are entered to change the shape and location of the I and C curves.
- A table of possibly relevant constants, such as e (Napier's constant) and T (for time, or T(w), which in these cases is always a constant = 1).
- A table of slopes of the I and C curves near relevant points. This is not always of interest.
- A data table for ICB line graphs with a header listing each data series:
  - W = number of workers, going from 0 to 99. This is the independent variable that is varied and is the basis for calculating income and costs.
  - I = the number of dollars of income associated with this number of workers.
  - C = the number of dollars of cost associated with this number of workers.
  - B = (I-C) = the number of dollars of benefits (or profits) associated with this number of workers.
  - P = (B/T) = the power of the benefits in dollars per day.
- A table for a scatter graph of power index ($\zeta=P/P_{max}$) versus efficiency (η) listing each variable.
- A table for a scatter graph of return on investment (ROI=B/C) versus power index (ζ) listing each variable.
- A table for a scatter graph of return on investment (ROI=B/C) versus efficiency (η) listing each variable.
- The first few elements of the data series for each variable in the above tables.

Graphs – There are six standard graphs in most of the annexes, as follows:

- Top left – An ICB curve set. That is, a line graph showing income (I), costs (C), and benefits (B).
- Top right – A $\eta\zeta$R curve set. That is, a line graph showing efficiency (η), power index (ζ), and return on investments (R).
- Middle left – A power index vs efficiency scatter graph in which I plot all calculated points in the table, including those that are patently nonsensical. For example, neither power index nor efficiency can be negative in the real world, but those values can be produced mathematically. I want to see the hidden shape of that part of the curve.
- Middle right – A power index vs efficiency scatter graph in which I plot only those points that have real-world interpretation. That is, I restrict the graph to the first quadrant only.
- Bottom left – A power index vs ROI scatter graph, restricted to the first quadrant.
- Bottom right – An ROI vs efficiency scatter graph, restricted to the first quadrant.

7.1 *Annex – Case A0*

Case A0 – Control Panel

Goldilocks Curves - Various Scenarios - CASE A0

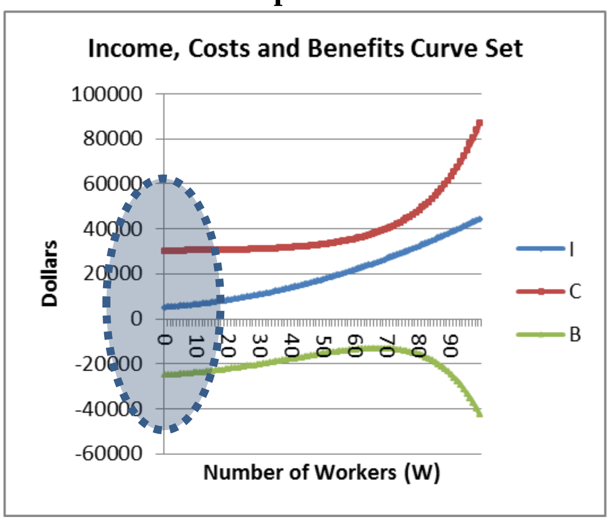
Parabola and Exponential

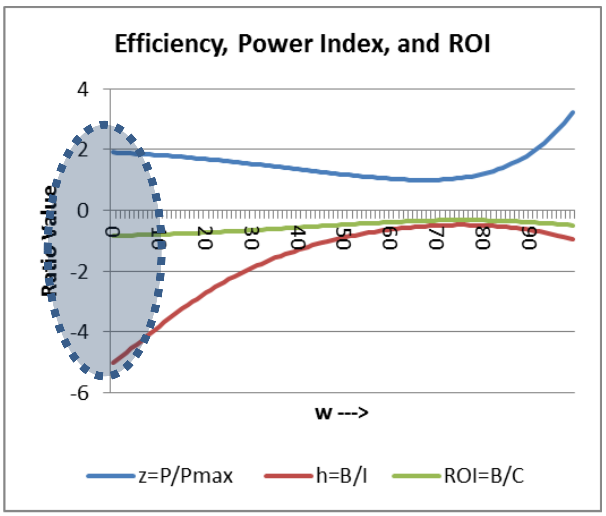
Control Panel

I=Parabola		C=Exponential			Constants		Intercepts at w=0		Slopes at w=0	
$I = aW^2 + bW + c$		$C = d + fe^{gW}$					$\alpha_I =$	5000	$\beta_I =$	103
a	3	d	30000		e	2.71828	$\alpha_C =$	30150	$\beta_C =$	9.275482
b	100	f	150		T	1	$(\alpha_I - \alpha_C)/\alpha_I =$		$(\beta_I - \beta_C)/\beta_I =$	
c	5000	g	0.06					-5.03		0.909947
					P_{max}	-13188.2				
			B=I-C	P=B/T	η=B/I	$\zeta = P/P_{max}$	ROI=B/C	$\zeta = P/P_{max}$	η=B/I	ROI=B/C
W	**I**	**C**	**B**	**P**	**η**	**ζ**	**R**	**ζ**	**η**	**R**
0	5000	30150	-25150	-25150	-5.03	1.90701	-0.83416	1.90701	-5.03	-0.834163
1	5103	30159.3	-25056.3	-25056.3	-4.91011	1.89991	-0.8308	1.89991	-4.91011	-0.830798
2	5212	30169.1	-24957.1	-24957.1	-4.7884	1.89239	-0.82724	1.89239	-4.7884	-0.827241

I have decided, entirely arbitrarily, to exhibit this case using a parabola and an exponential curve.

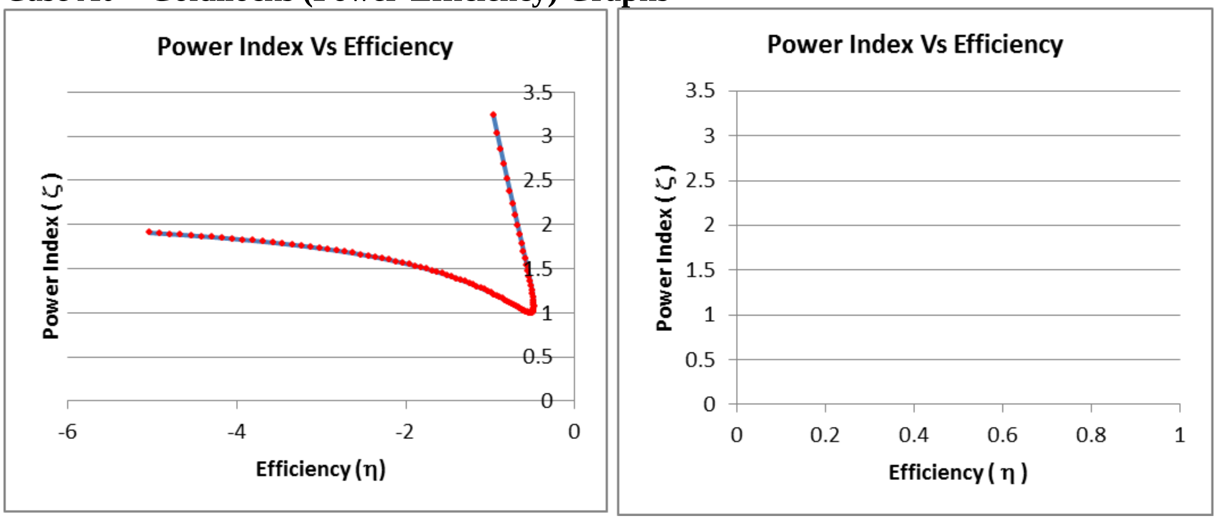
Case A0 – Line Graphs





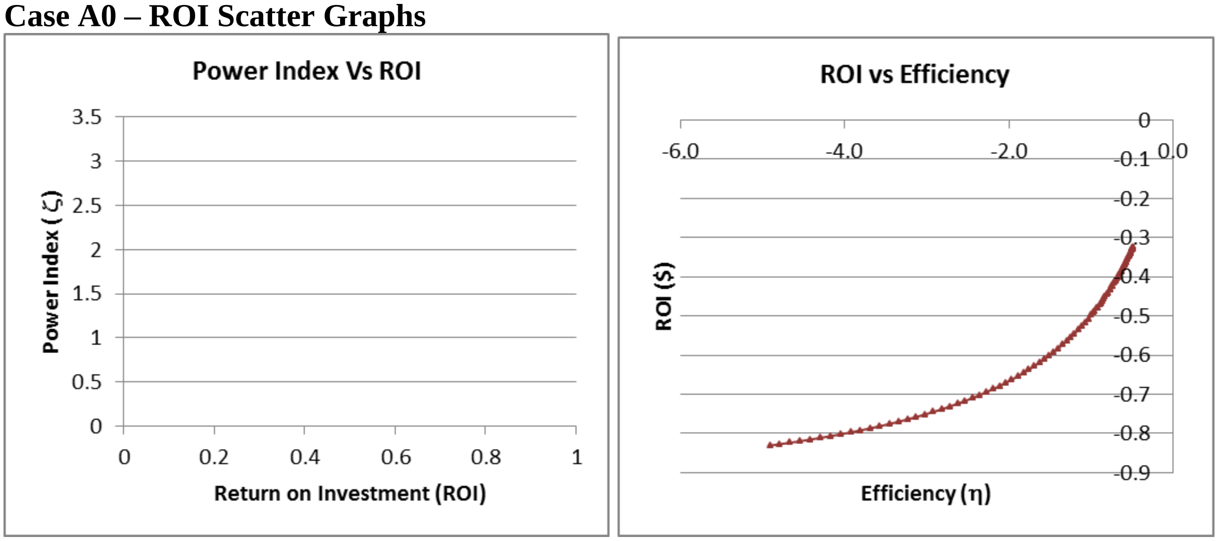
This would be an example of an ICBT curve set that is excluded from Π. Case A focuses on the part of the graph where w = 0, as circled. But, throughout the domain of w, the income never exceeds the cost, and so this curve set is NOT in Π. It is shown here as an example for which C(0) > I(0). It is excluded from Π because this business model fails to make a profit for all values of the parameter w, and so can not be indicative of a persistent business activity. I.e. it does not meet Cond 05. It is meaningless to calculate either the power of the benefits, or the efficiency of the business activity for the parameter value w = 0, in this circumstance. There are several other examples of ICBT curve sets having the Case A0 characteristic which have arisen while examining other characteristics, and which are included in **Π.**

Case A0 – Goldilocks (Power-Efficiency) Graphs



Since the graph on the right is restricted to the 1st quadrant, and the graph on the left, from which it is taken, never enters the 1st quadrant, the right-hand graph is blank.

Case A0 – ROI Scatter Graphs



These graphs are meaningless, but included for completeness.

7.2 Annex – Case A1a

Case A1a – Control Panel

Goldilocks Curves - CASE A1a

Parabola and Exponential

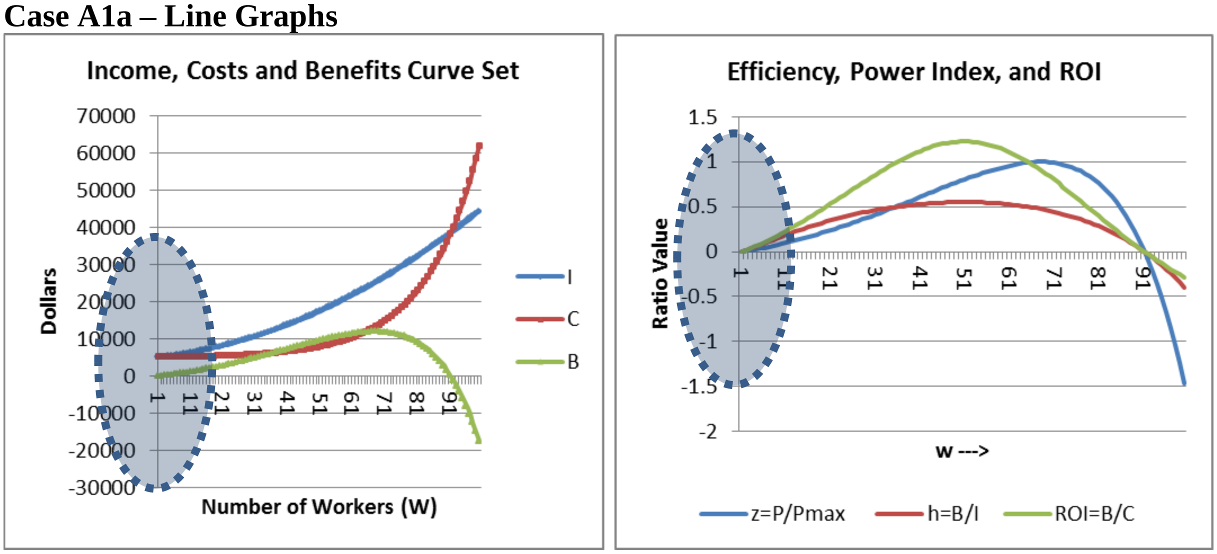
Control Panel

I=Parabola		C=Exponential			Constants		Intercepts at w=0		Slopes at w=0	
$I = aW^2 + bW + c$		$C = d + fe^{gW}$					α_I =	5000	β_I =	103
a	3	d	4850		e	2.718282	α_C =	5000	β_C =	9.275482
b	100	f	150		T	1	$(\alpha_I - \alpha_C)/\alpha_I$ =		$(\beta_I - \beta_C)/\beta_I$ =	
c	5000	g	0.06					0		0.9099468
					P_{max}	11961.83				
			B=I-C	P=B/T	η=B/I	$\zeta = P/P_{max}$	ROI=B/C	$\zeta = P/P_{max}$	η=B/I	ROI=B/C
W	**I**	**C**	**B**	**P**	**η**	**ζ**	**ROI**	**ζ**	**η**	**ROI**
0	5000	5000	0	0	0	0	0	0	0	0
1	5103	5009.275	93.72452	93.72452	0.018367	0.007835	0.01871	0.0078353	0.018367	0.0187102
2	5212	5019.125	192.8755	192.8755	0.037006	0.016124	0.038428	0.01612424	0.037006	0.0384281

I have arbitrarily decided to use a parabola and an exponential curve to demonstrate this characteristic.

Focus on w=0. I(0)>0; C(0)=I(0); B(0)=0; P(0)=0; η(0)=0; R(0)=0.

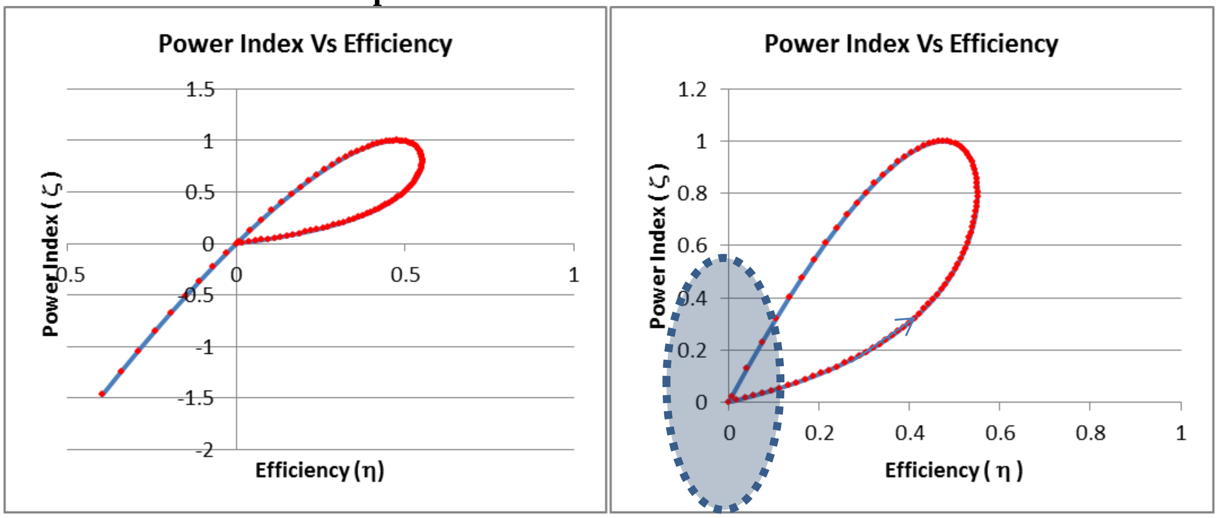
Case A1a – Line Graphs



The curve set is an element of Π.

The focus is on w=0. I(0)>0. C(0)=I(0). B(0) = 0.

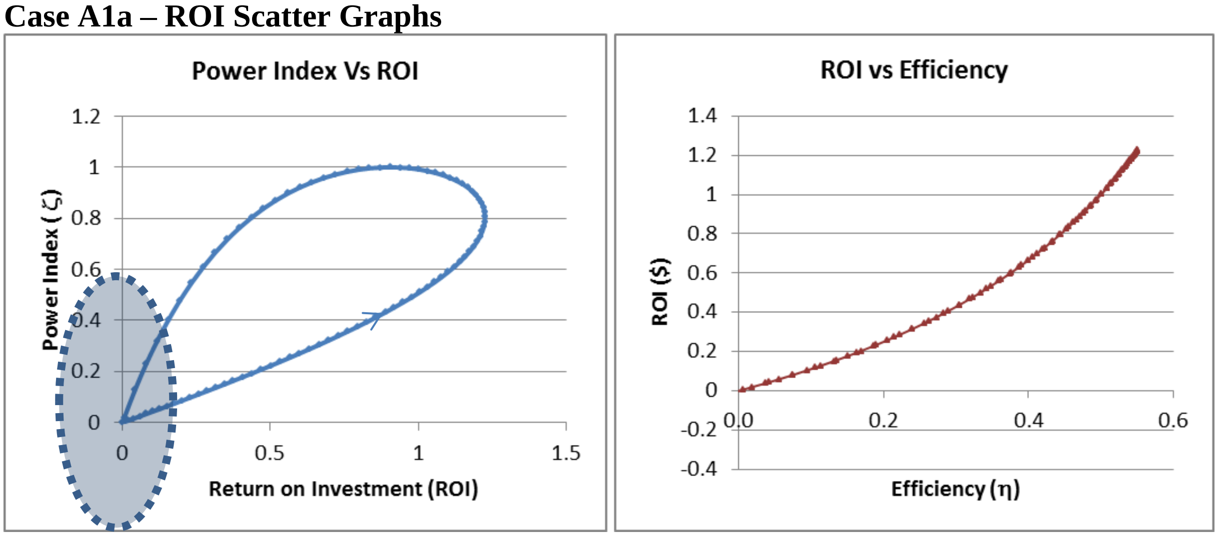
Case A1a – Goldilocks Graphs



Some of the data points are meaningless, falling elsewhere than in the 1st quadrant. The meaningful points are graphed to the right. The shaded oval indicates the area of interest in the graphs.

Focus on w = 0. ζ(0) = 0. η(0) = 0.

Case A1a – ROI Scatter Graphs



Only the 1st quadrant is included in the graphs.

Focus on w = 0. ζ(0) = 0. η(0) = 0. R(0) = 0. $R = \eta/(1-\eta)$.

7.3 Annex – Case A1b

Case A1b – Control Panel

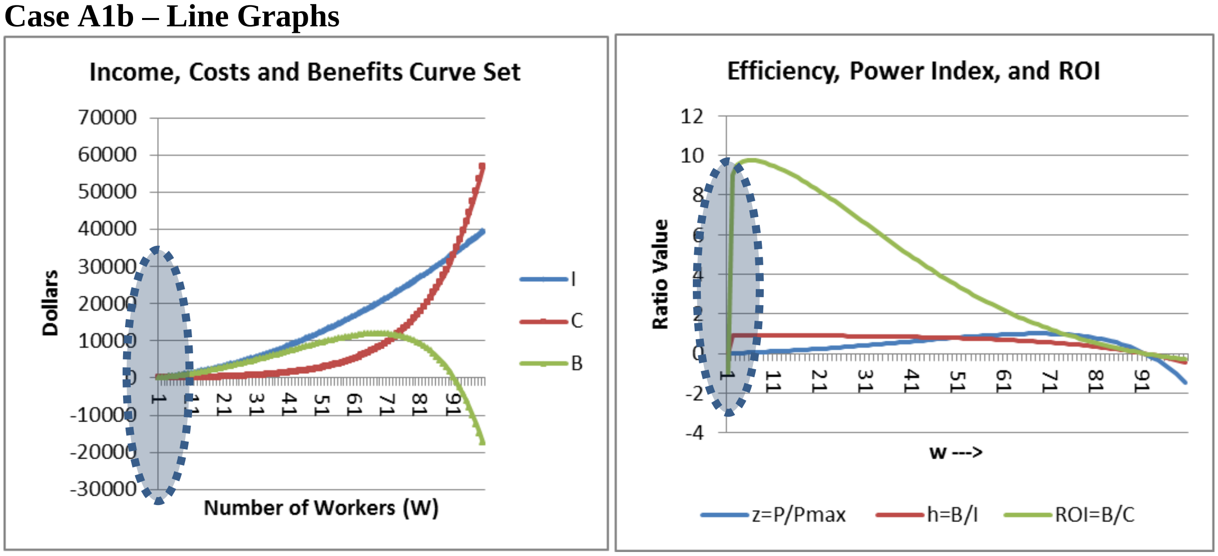
Goldilocks Curves - CASE A1b

Parabola and Exponential

Control Panel										
I=Parabola		C=Exponential			Constants		Intercepts at w=0		Slopes at w=0	
$I = aW^2 + bW + c$		$C = d + fe^{gW}$								
a	3	d	-149		e	2.71828	α_I =	0	β_I =	103
b	100	f	150		T	1	α_C =	1	β_C =	9.27548
c	0	g	0.06				$(\alpha_I - \alpha_C)/\alpha_I$ =		$(\beta_I - \beta_C)/\beta_I$ =	
								#DIV/0!		0.90995
					P_{max}	11960.8				
			B=I-C	P=B/T	η=B/I	$\zeta=P/P_{max}$	ROI=B/C	$\zeta=P/P_{max}$	η=B/I	ROI=B/C
W	I	C	B	P	η	ζ	ROI	ζ	η	ROI
0	0	1	-1	-1	#DIV/0!	-8.4E-05	-1	-8.4E-05	#DIV/0!	-1
1	103	10.2755	92.7245	92.7245	0.90024	0.00775	9.02386	0.00775	0.90024	9.02386
2	212	20.1245	191.875	191.875	0.90507	0.01604	9.53441	0.01604	0.90507	9.53441

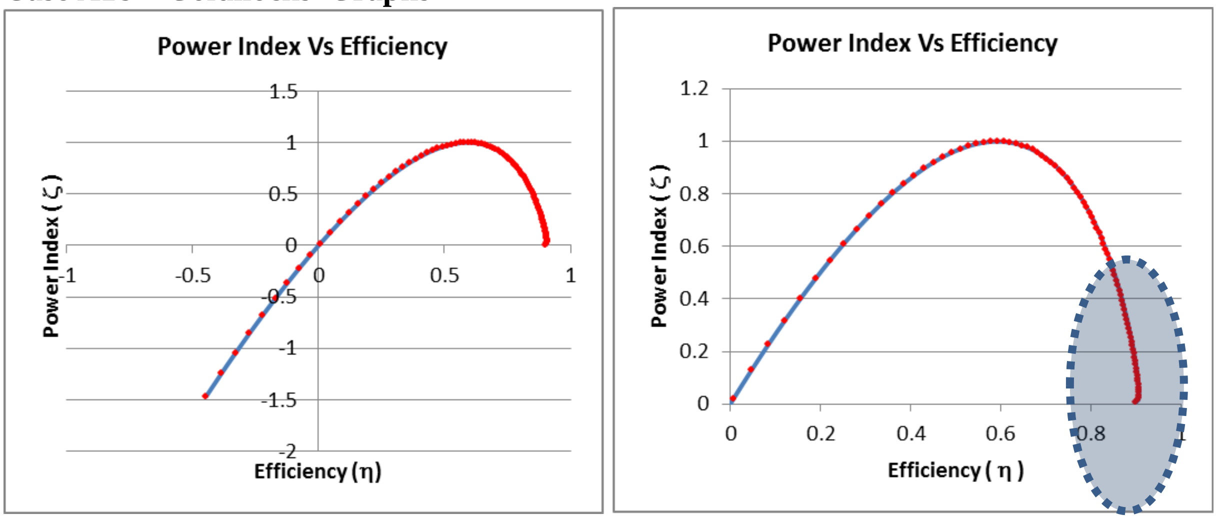
I have arbitrarily decided to use a parabola and an exponential curve to demonstrate this characteristic.

Case A1b – Line Graphs



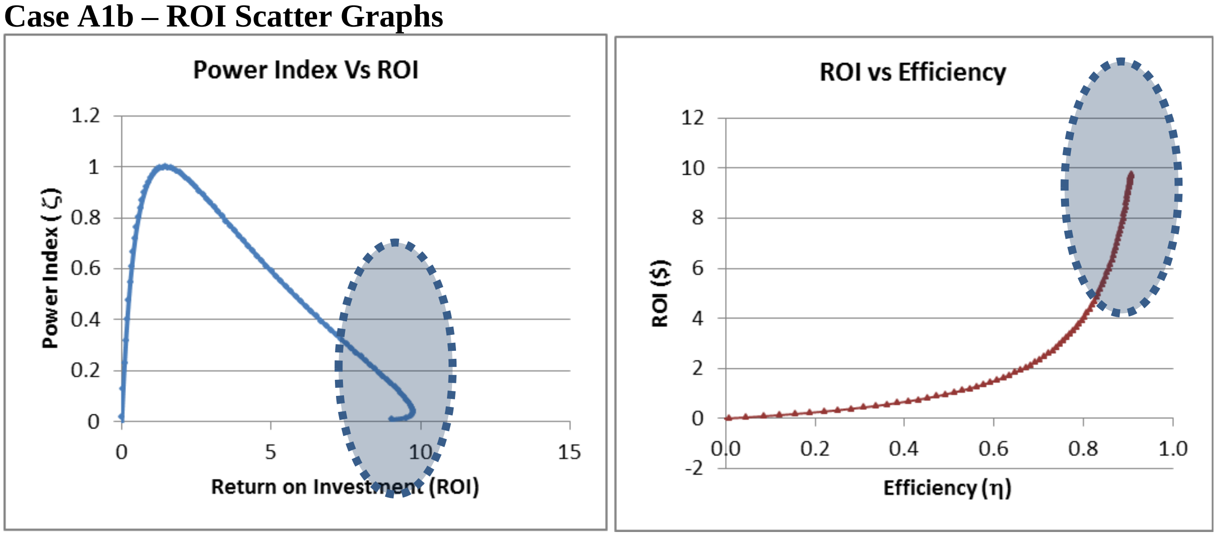
The curve set is an element of Π.
The focus is on w=0. I(0)=0. C(0)=I(0). B(0) = 0.

Case A1b – Goldilocks Graphs



Some of the data points are meaningless, falling elsewhere than in the 1st quadrant. The meaningful points are graphed to the right.   The shaded oval indicates the area of interest in the graphs.
Focus on w = 0. ζ(0) = 0. η(0) $= (\beta_I - \beta_C)/\beta_I$.

Case A1b – ROI Scatter Graphs



Only the 1st quadrant is included in the graphs.
Focus on w = 0. ζ(0) = 0. η(0) = 0/0 = $(\beta_I - \beta_C)/\beta_I$. R(0) = 0/0 = $(\beta_I - \beta_C)/\beta_C = \eta/(1-\eta)$.

7.4 Annex – Case A2a

Case A2a – Control Panel

Goldilocks Curves - CASE A2a

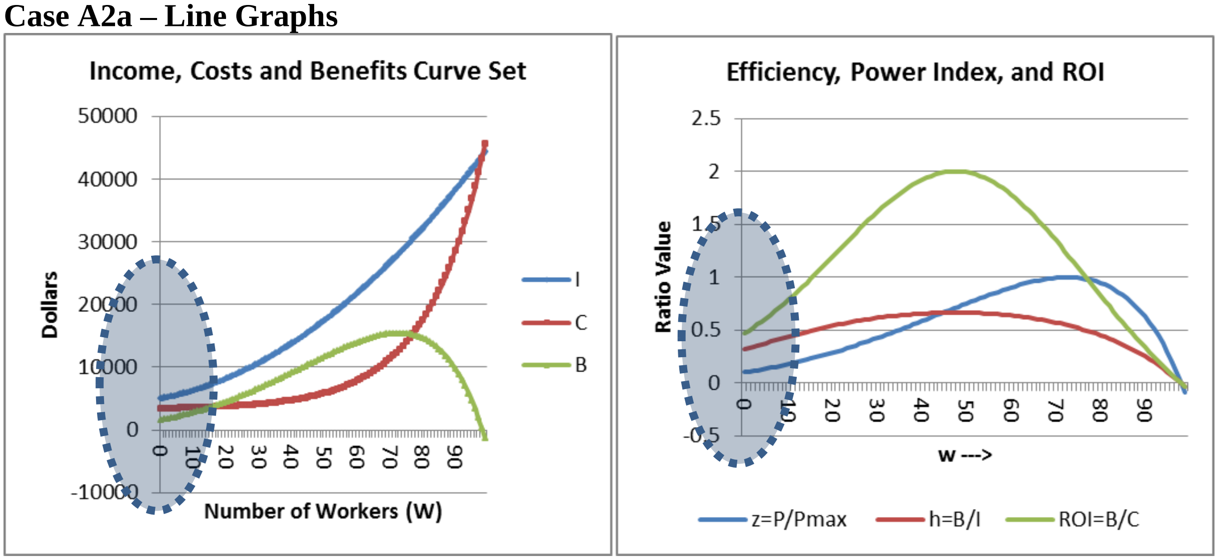
Parabola and Exponential

Control Panel

I=Parabola		**C=Exponential**			**Constants**		**Intercepts at w=0**		**Slopes at w=0**	
$I = aW^2 + bW + c$		$C = d + fe^{gW}$					α_I =	5000	β_I =	103
a	3	d	3250		e	2.718282	α_C =	3400	β_C =	8.798372
b	100	f	150		T	1	$(\alpha_I - \alpha_C)/\alpha_I$ =		$(\beta_I - \beta_C)/\beta_I$ =	
c	5000	g	0.057					0.32		0.914579
					P_{max}	15416.66				
			B=I-C	P=B/T	η=B/I	$\zeta = P/P_{max}$	ROI=B/C	$\zeta = P/P_{max}$	η=B/I	ROI=B/C
W	**I**	**C**	**B**	**P**	**η**	**ζ**	**ROI**	**ζ**	**η**	**ROI**
0	5000	3400	1600	1600	0.32	0.103784	0.470588	0.103784	0.32	0.470588
1	5103	3408.798	1694.202	1694.202	0.332001	0.109894	0.497008	0.109894	0.332001	0.497008
2	5212	3418.113	1793.887	1793.887	0.344184	0.11636	0.524818	0.11636	0.344184	0.524818

I have arbitrarily decided to use a parabola and an exponential curve to demonstrate this characteristic.

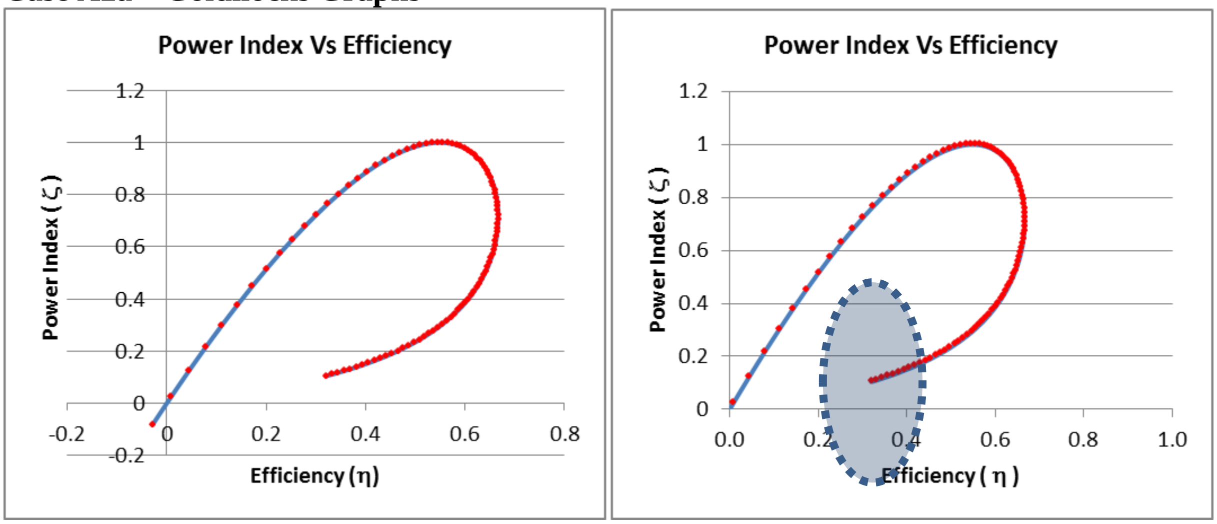
Case A2a – Line Graphs



The curve set is an element of Π.

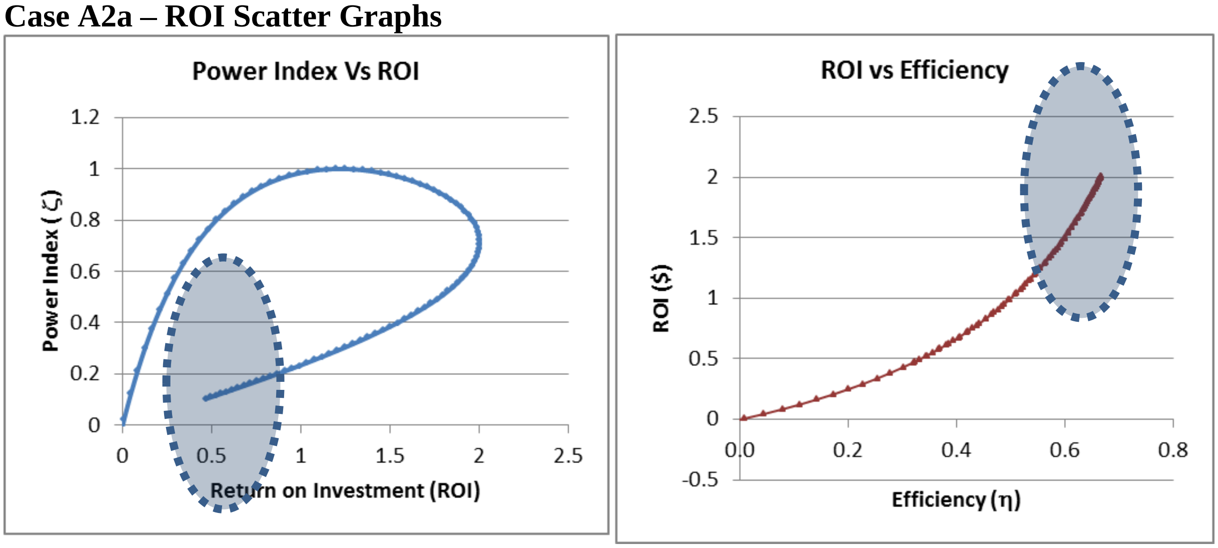
The focus is on w=0. I(0)>0. C(0)>0. B(0) > 0. w_L, if it exists mathematically, would be a virtual lower bound to the left of w = 0.

Case A2a – Goldilocks Graphs



Some of the data points are meaningless, falling elsewhere than in the 1st quadrant. The meaningful points are graphed to the right. The shaded oval indicates the area of interest in the graphs. I note that this forms part of a loop that could be closed at the origin.
Focus on w = 0. $\zeta(0) > 0$. $\eta(0) = (\alpha_I - \alpha_C)/\alpha_I$.

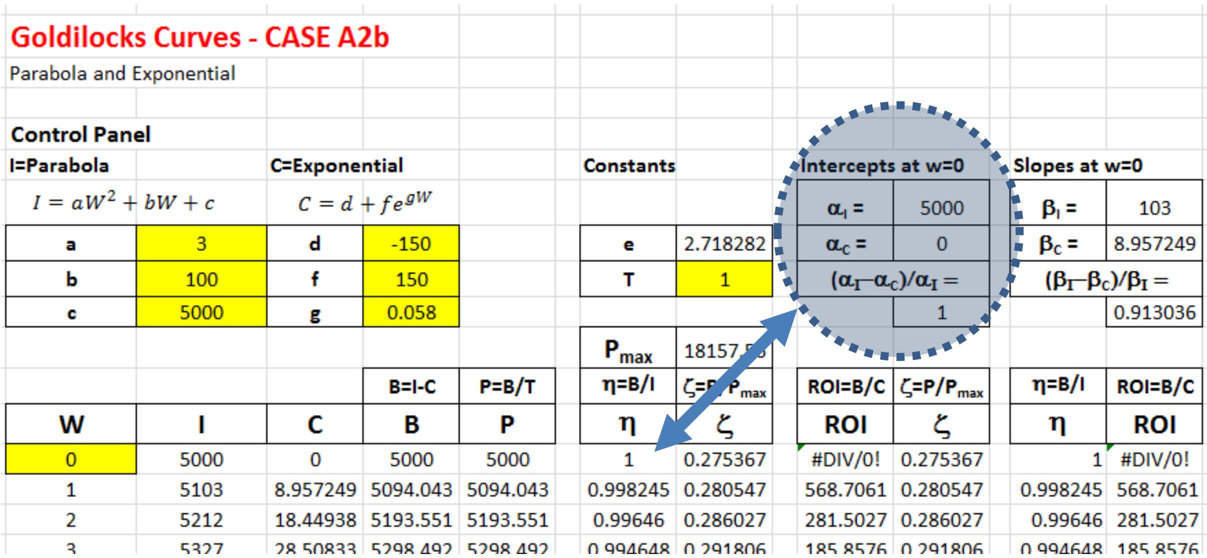
Case A2a – ROI Scatter Graphs



Only the 1st quadrant is included in the graphs.
Focus on w = 0. $\zeta(0) > 0$. $\eta(0) = (\alpha_I - \alpha_C)/\alpha_I$. $R(0) = (\alpha_I - \alpha_C)/\alpha_C = \eta/(1-\eta)$.

7.5 Annex – Case A2b

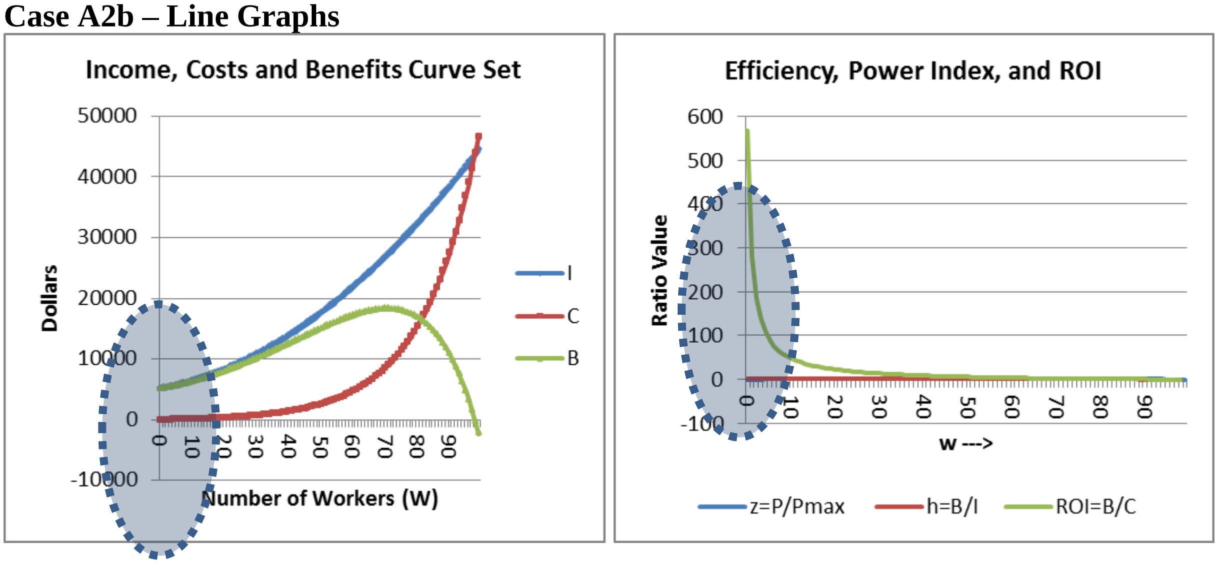
Case A2b – Control Panel



Goldilocks Curves - CASE A2b										
Parabola and Exponential										
Control Panel										
I=Parabola		C=Exponential			Constants		Intercepts at w=0		Slopes at w=0	
$I = aW^2 + bW + c$		$C = d + fe^{gW}$					α_I =	5000	β_I =	103
a	3	d	-150		e	2.718282	α_C =	0	β_C =	8.957249
b	100	f	150		T	1	$(\alpha_I-\alpha_C)/\alpha_I$ =		$(\beta_I-\beta_C)/\beta_I$ =	
c	5000	g	0.058					1		0.913036
					P_{max}	18157.5				
			B=I-C	P=B/T	η=B/I	ζ=P/P_{max}	ROI=B/C	ζ=P/P_{max}	η=B/I	ROI=B/C
W	I	C	B	P	η	ζ	ROI	ζ	η	ROI
0	5000	0	5000	5000	1	0.275367	#DIV/0!	0.275367	1	#DIV/0!
1	5103	8.957249	5094.043	5094.043	0.998245	0.280547	568.7061	0.280547	0.998245	568.7061
2	5212	18.44938	5193.551	5193.551	0.99646	0.286027	281.5027	0.286027	0.99646	281.5027
3	5327	28.50833	5298.492	5298.492	0.994648	0.291806	185.8576	0.291806	0.994648	185.8576

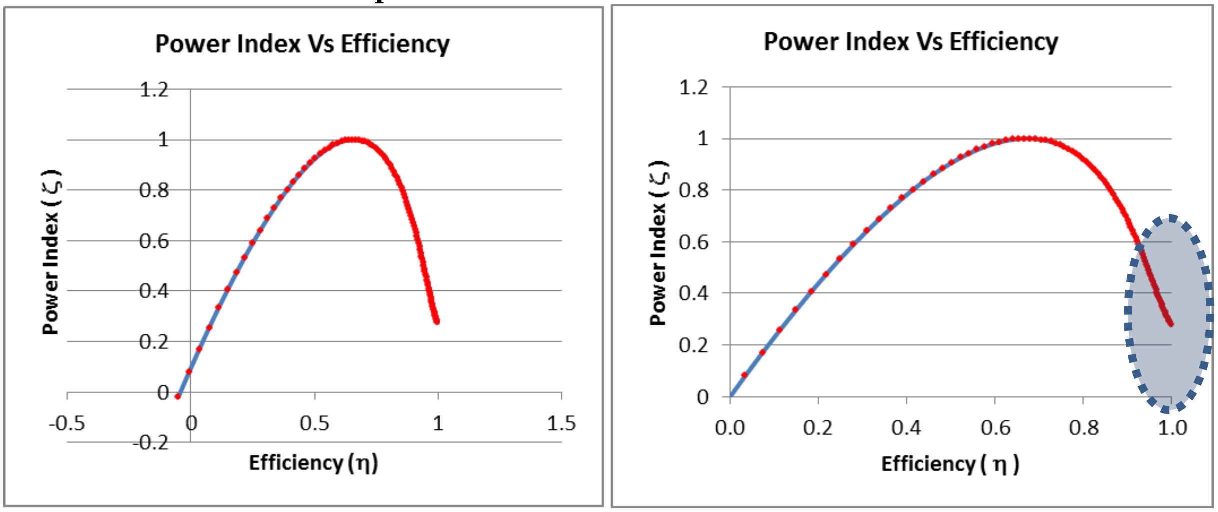
I have arbitrarily decided to use a parabola and an exponential curve to demonstrate this characteristic.

Case A2b – Line Graphs



The curve set is an element of Π.
The focus is on w=0. I(0)= B(0)>0. C(0)=0. w_L, the lower bound of the interval on which B(w) is positive, if it exists, is virtual, and is located to the left of w=0.

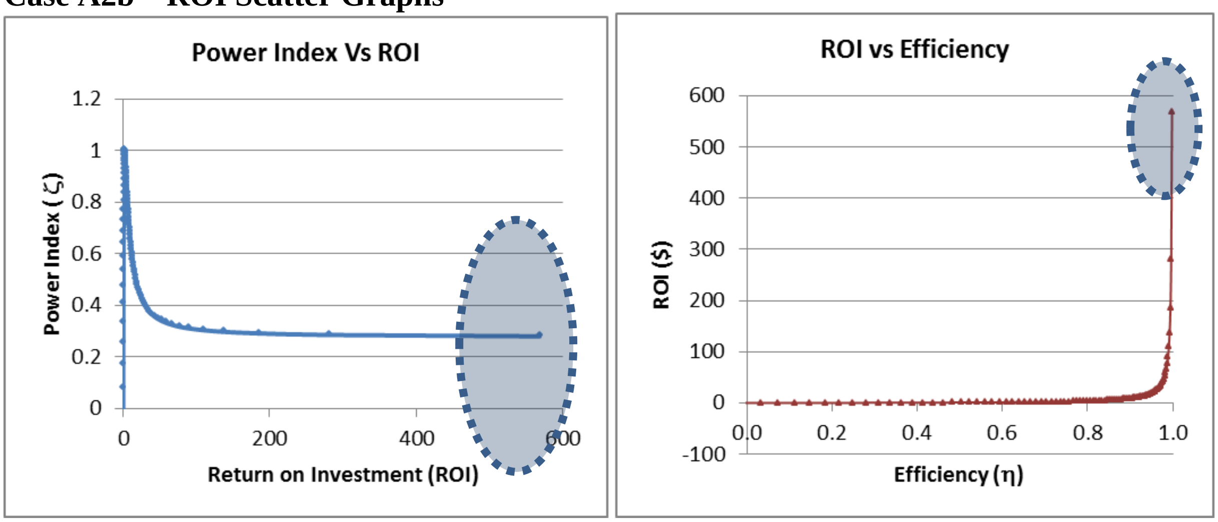
Case A2b – Goldilocks Graphs



Some of the data points are meaningless, falling elsewhere than in the 1st quadrant. The meaningful points are graphed to the right. The shaded oval indicates the area of interest in the graphs. Note that this is no longer a loop. Compare with case A2a.

Focus on w = 0. $\zeta(0) > 0$. $\eta(0) = (\alpha_I - \alpha_C)/\alpha_I$.

Case A2b – ROI Scatter Graphs



Only the 1st quadrant is included in the graphs.

Focus on w = 0. $\zeta(0) > 0$. $\eta(0) = (\alpha_I - \alpha_C)/\alpha_I$. $R(0) = (\alpha_I - \alpha_C)/\alpha_C = \eta/(1-\eta)$.

7.6 Annex – Case B1

Case B1 – Control Panel

Goldilocks Curves - CASE B1

Parabola and Exponential

1 For case B1, set c to 0, then copy D12 to D11.

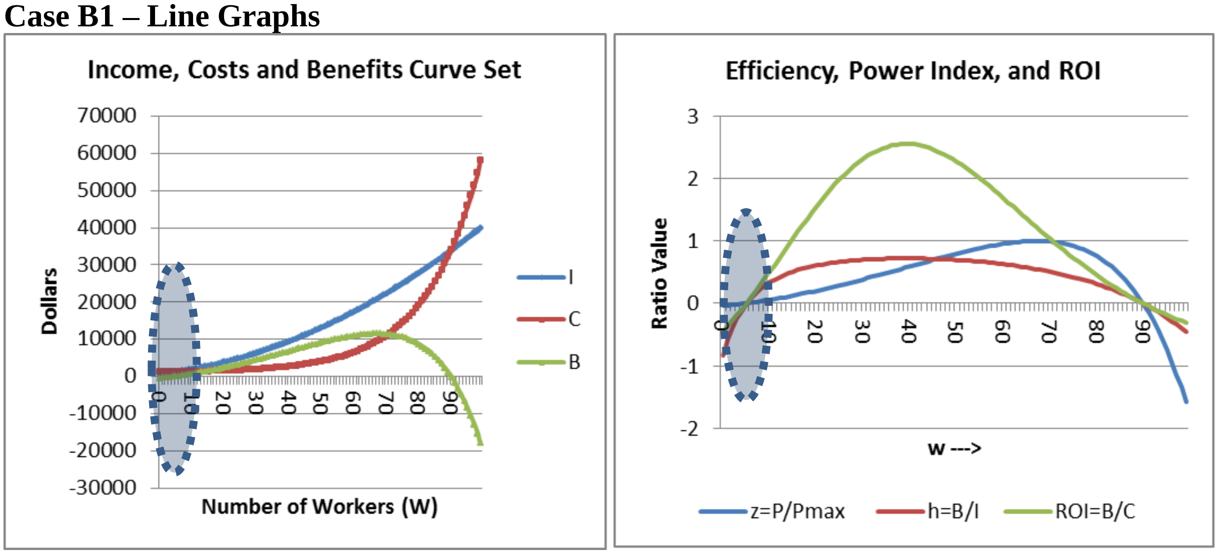
Control Panel

I=Parabola		C=Exponential		Constants		Intercepts at w_L=5		Slopes at w_L=5	
$I = aW^2 + bW + c$		$C = d + fe^{gW}$				α_I =	537.4788	β_I =	133
a	3	d	1000	e	2.718282	α_C =	1139.876	β_C =	12.52059
b	100	f	150	T	1	$(\alpha_I - \alpha_C)/\alpha_I$ =		$(\beta_I - \beta_C)/\beta_I$ =	
c	627.4788211	g	0.06				-0.83273		0.90586
w=5 c parm	0			P_{max}	11439.31				

			B=I-C	P=B/T	η=B/I	ζ=P/P_{max}	ROI=B/C	ζ=P/P_{max}	η=B/I	ROI=B/C
W	**I**	**C**	**B**	**P**	**η**	**ζ**	**ROI**	**ζ**	**η**	**ROI**
0	627.4788211	1150	-522.521	-522.521	-0.83273	-0.04568	-0.45437	-0.04568	-0.83273	-0.45437
1	730.4788211	1159.275	-428.797	-428.797	-0.58701	-0.03748	-0.36988	-0.03748	-0.58701	-0.36988
2	839.4788211	1169.125	-329.646	-329.646	-0.39268	-0.02882	-0.28196	-0.02882	-0.39268	-0.28196
3	954.4788211	1179.583	-225.104	-225.104	-0.23584	-0.01968	-0.19083	-0.01968	-0.23584	-0.19083
4	1075.478821	1190.687	-115.209	-115.209	-0.10712	-0.01007	-0.09676	-0.01007	-0.10712	-0.09676
5	1202.478821	1202.479	0	0	0	0	0	0	0	0
6	1335.478821	1214.999	120.4794	120.4794	0.090214	0.010532	0.09916	0.010532	0.090214	0.09916
7	1474.478821	1228.294	246.1846	246.1846	0.166964	0.021521	0.200428	0.021521	0.166964	0.200428
8	1619.478821	1242.411	377.0677	377.0677	0.232833	0.032962	0.303497	0.032962	0.232833	0.303497

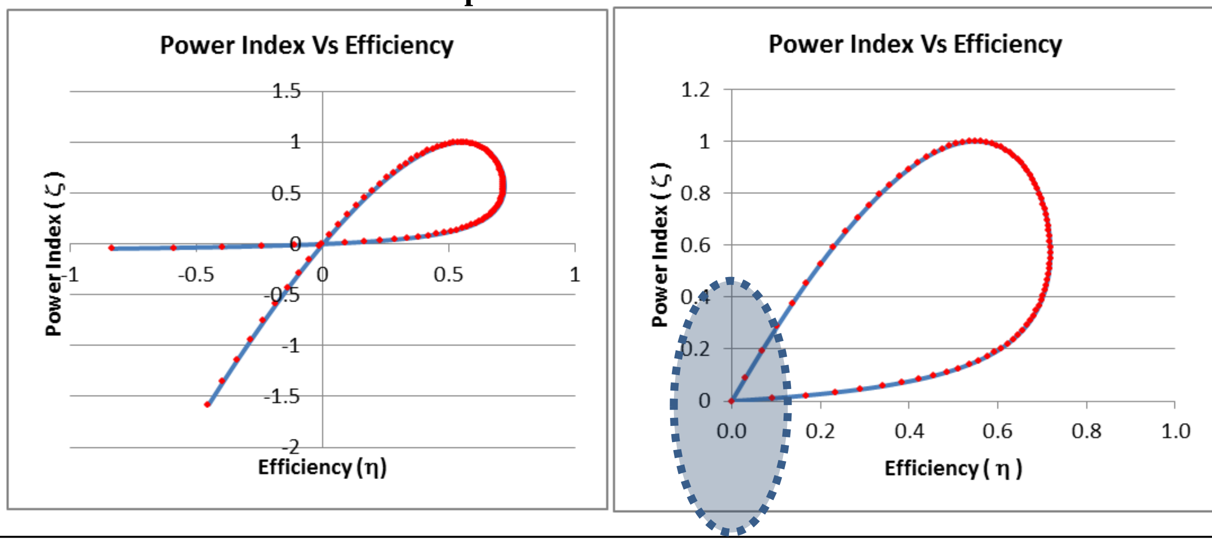
I have arbitrarily decided to use a parabola and an exponential curve to demonstrate this characteristic.

Case B1 – Line Graphs



The curve set is an element of Π.
The focus is on w=w_L=5. I(w_L)=C(w_L)>0. B(w_L) = 0. w_L, i.e. the lower bound of the interval on which B(w) is positive, is located at w=5.

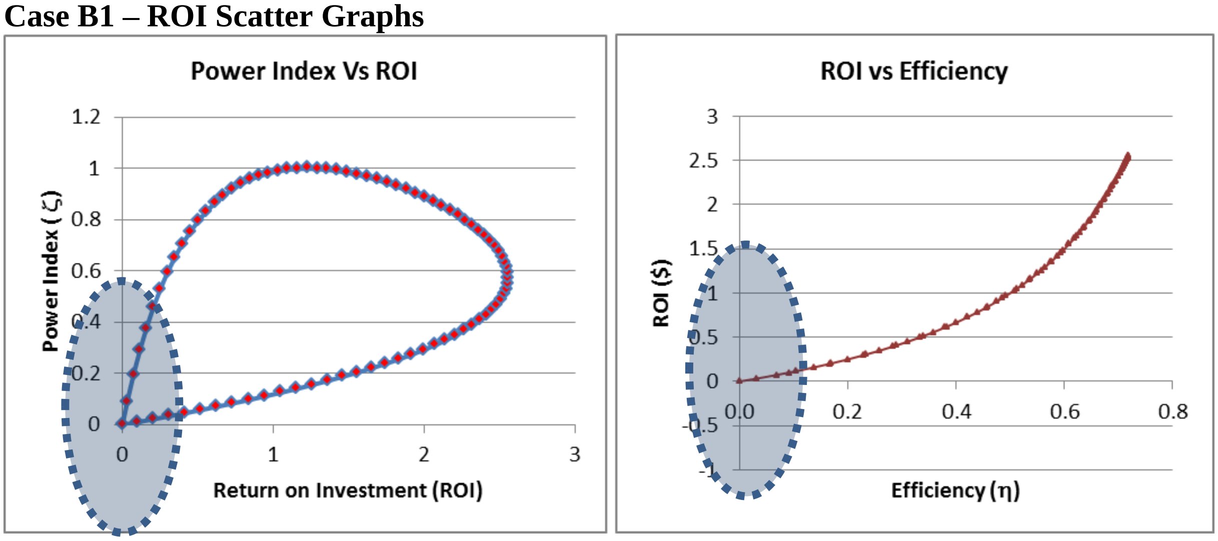
Case B1 – Goldilocks Scatter Graphs



Some of the data points are meaningless, falling elsewhere than in the 1st quadrant. The meaningful points are graphed to the right. The shaded oval indicates the area of interest in the graphs.

Focus on w = 0. $\zeta(w_L) = 0$. $\eta(w_L) = 0$.

Case B1 – ROI Scatter Graphs



Only the 1st quadrant is included in the graphs.

Focus on w = 0. $\zeta(w_L) = 0$. $\eta(w_L) = 0$. $R(w_L) = 0$.

7.7 Annex – Case B2

Case B2 – Control Panel

Goldilocks Curves - CASE B2

Parabola and Exponential

1 For case B2, set d to 0, then copy F12 to F9.

2 For case B2, then set c to 0, then copy D12 to D11.

Control Panel

I=Parabola		C=Exponential			Constants	
$I = aW^2 + bW + c$		$C = d + fe^{gW}$				
a	3	d	-202.4788		e	2.718282
b	100	f	150		T	1
c	-575	g	0.06			
w=5 c parm	0	w=5 d parm	0		P_{max}	11439.31

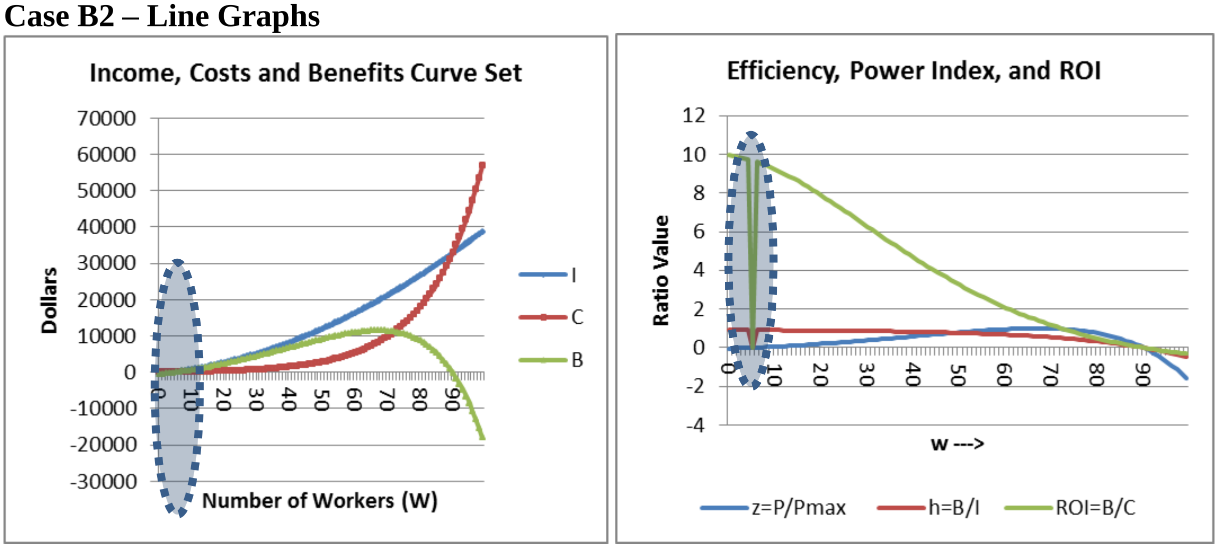
Intercepts at w_L=5	
α_I =	-665
α_C =	-62.603
$(\alpha_I - \alpha_C)/\alpha_I$ =	
	0.908732

Slopes at w_L=5	
β_I =	133
β_C =	12.52059
$(\beta_I - \beta_C)/\beta_I$ =	
	0.90586

			B=I-C	P=B/T	η=B/I	ζ=P/P_{max}	ROI=B/C	ζ=P/P_{max}	η=B/I	ROI=B/C
W	**I**	**C**	**B**	**P**	**η**	**ζ**	**ROI**	**ζ**	**η**	**ROI**
0	-575	-52.478821	-522.5212	-522.521	0.908732	-0.04568	9.956801	-0.04568	0.908732	9.956801
1	-472	-43.203339	-428.7967	-428.797	0.908468	-0.03748	9.925081	-0.03748	0.908468	9.925081
2	-363	-33.354293	-329.6457	-329.646	0.908115	-0.02882	9.883157	-0.02882	0.908115	9.883157
3	-248	-22.896217	-225.1038	-225.104	0.907677	-0.01968	9.831484	-0.01968	0.907677	9.831484
4	-127	-11.791449	-115.2086	-115.209	0.907154	-0.01007	9.770517	-0.01007	0.907154	9.770517
5	0	0	0	0	#DIV/0!	0	#DIV/0!	0	#DIV/0!	#DIV/0!
6	133	12.520591	120.47941	120.4794	0.90586	0.010532	9.622502	0.010532	0.90586	9.622502
7	272	25.815412	246.18459	246.1846	0.90509	0.021521	9.536342	0.021521	0.90509	9.536342
8	417	39.932339	377.06766	377.0677	0.904239	0.032962	9.442664	0.032962	0.904239	9.442664

I have arbitrarily decided to use a parabola and an exponential curve to demonstrate this characteristic.

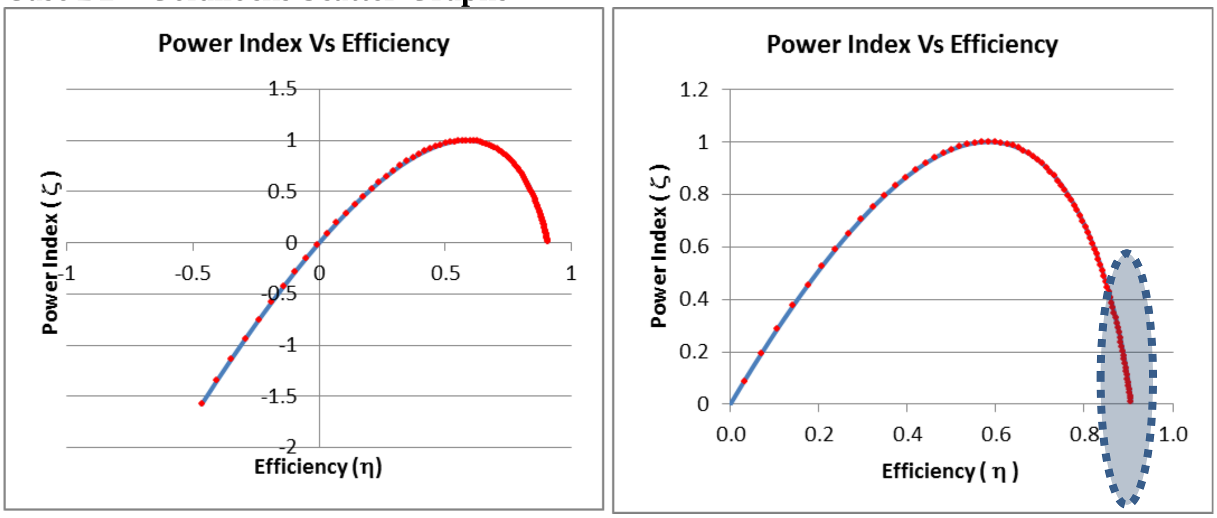
Case B2 – Line Graphs



The curve set is an element of Π.

The focus is on w=w_L=5. I(w_L)=0. C(w_L)=0. B(w_L) = 0. w_L, the lower bound of the interval on which B(w) is positive, is located at w=5. η(w_L) and R(w_L) are indeterminate.

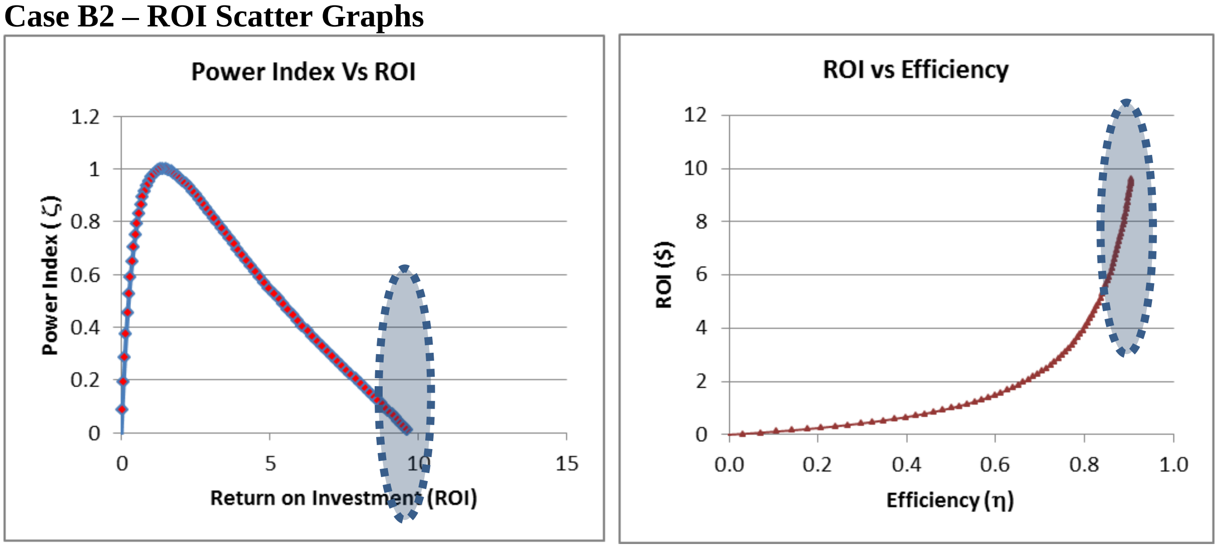
Case B2 – Goldilocks Scatter Graphs



Some of the data points are meaningless, falling elsewhere than in the 1st quadrant. The meaningful points are graphed to the right. The shaded oval indicates the area of interest in the graphs.

Focus on w = 0. $\zeta(0) = 0$. $\eta(w_L) = (\beta_I - \beta_C)/\beta_I$.

Case B2 – ROI Scatter Graphs



Only the 1st quadrant is included in the graphs.

Focus on w = 0. $\zeta(0) = 0$. $\eta(w_L) = (\beta_I - \beta_C)/\beta_I$. $R(w_L) = (\beta_I - \beta_C)/\beta_C = \eta/(1 - \eta)$.

7.8 Annex – Case C1

Case C1 – Control Panel

Goldilocks Curves - CASE C1

Parabola and Parabola

1 For case C1, set c to 0, then copy D12 to D11.

Control Panel

I=Parabola		C=Parabola		Constants		Intercepts at w_U=80		Slopes at w_U=80	
$I = a(w-b)^2 + c$		$C = d(w-f)^2 + g$				α_I =	25925	β_I =	-295
a	-5	d	-3	e	2.71828	α_C =	18885	β_C =	-207
b	50	f	45	T	1	$(\alpha_I-\alpha_C)/\alpha_I$ =		$(\beta_I-\beta_C)/\beta_I$ =	
c	6825	g	6000				0.98678		0.29831
w=80 c parm	0			P_{max}	1012				

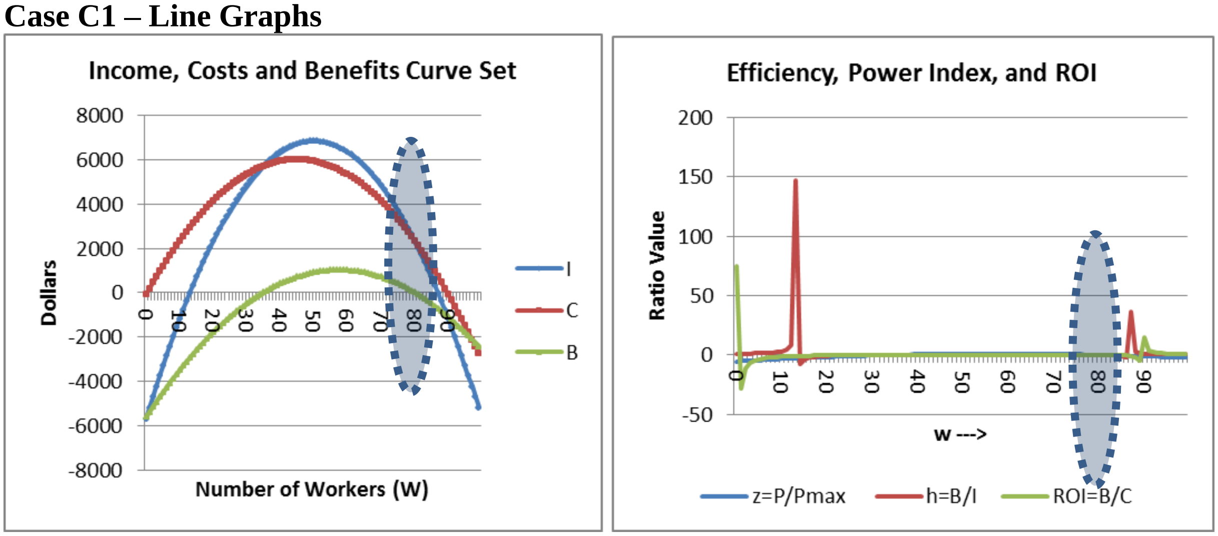
			B=I-C	P=B/T	η=B/I	ζ=P/P_{max}	ROI=B/C	ζ=P/P_{max}	η=B/I	ROI=B/C
W	I	C	B	P	η	ζ	ROI	ζ	η	ROI
0	-5675	-75	-5600	-5600	0.98678	-5.5336	74.6667	-5.5336	0.98678	74.6667
1	-5180	192	-5372	-5372	1.03707	-5.3083	-27.9792	-5.3083	1.03707	-27.9792
2	-4695	453	-5148	-5148	1.09649	-5.08696	-11.3642	-5.08696	1.09649	-11.3642

And, further down ---

76	3445	3117	328	328	0.09521	0.32411	0.10523	0.32411	0.09521	0.10523
77	3180	2928	252	252	0.07925	0.24901	0.08607	0.24901	0.07925	0.08607
78	2905	2733	172	172	0.05921	0.16996	0.06293	0.16996	0.05921	0.06293
79	2620	2532	88	88	0.03359	0.08696	0.03476	0.08696	0.03359	0.03476
80	2325	2325	0	0	0	0	0	0	0	0
81	2020	2112	-92	-92	-0.04554	-0.09091	-0.04356	-0.09091	-0.04554	-0.04356
82	1705	1893	-188	-188	-0.11026	-0.18577	-0.09931	-0.18577	-0.11026	-0.09931
83	1380	1668	-288	-288	-0.2087	-0.28458	-0.17266	-0.28458	-0.2087	-0.17266

I have arbitrarily decided to use a parabola and an exponential curve to demonstrate this characteristic.

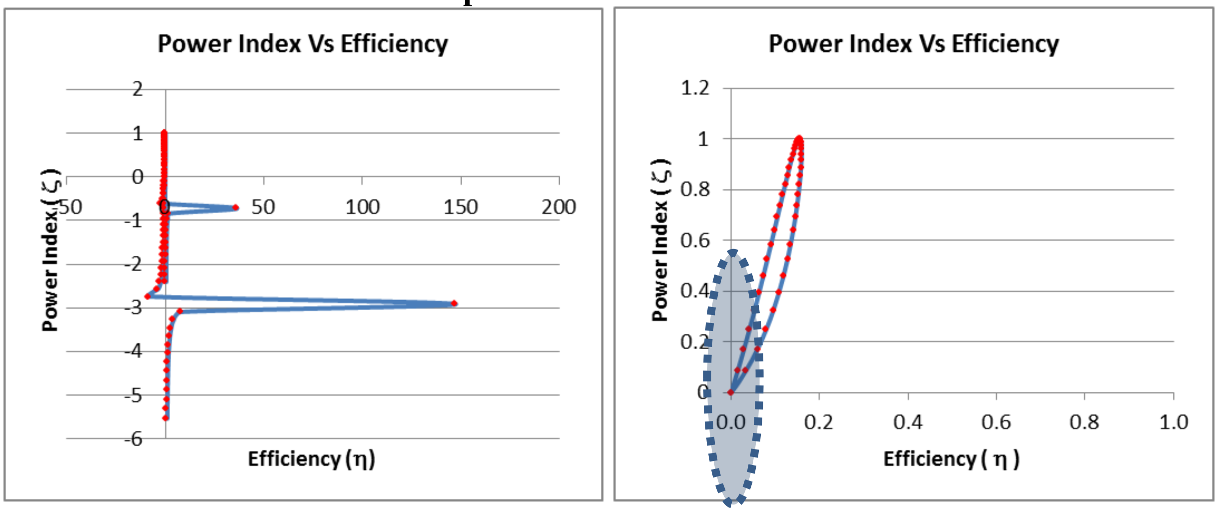
Case C1 – Line Graphs



The curve set is an element of Π.

The focus is on $w=w_U=80$. $I(w_U)>0$. $C(w_U)=I(w_U)$. $B(w_U) = 0$. w_U, the upper bound of the interval on which B(w) is positive, is located at w=80.

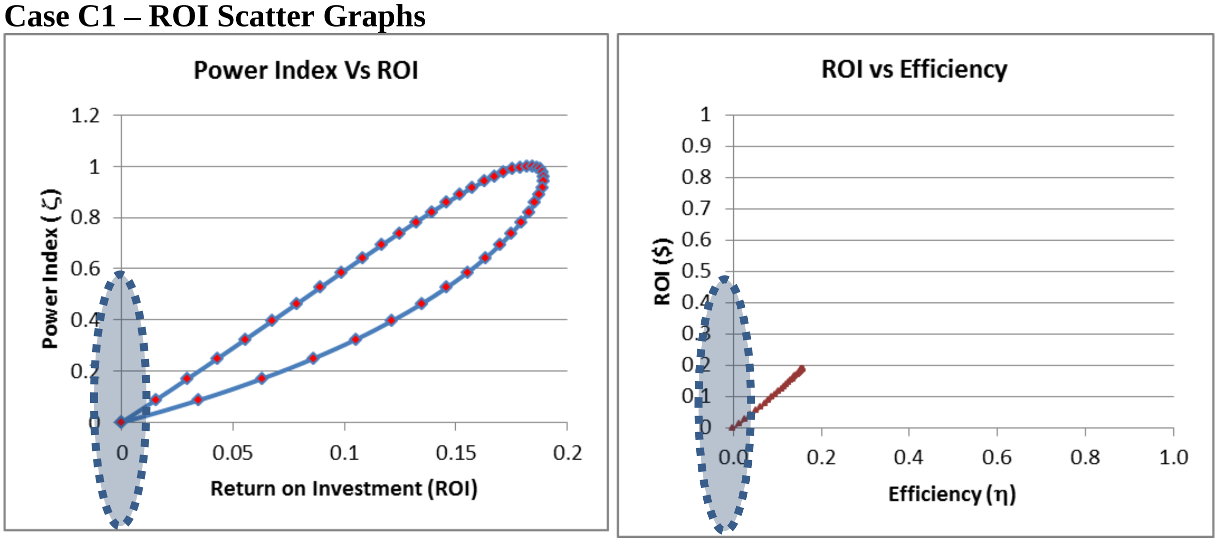
Case C1 – Goldilocks Scatter Graphs



Some of the data points are meaningless, falling elsewhere than in the 1st quadrant. The meaningful points are graphed to the right. The shaded oval indicates the area of interest in the graphs.

Focus on w = 80. $\zeta(w_U) = 0$. $\eta(w_U) = 0$.

Case C1 – ROI Scatter Graphs



Only the 1st quadrant is included in the graphs.

Focus on w = 80. $\zeta(w_U) = 0$. $\eta(w_U) = 0$. $R(w_U) = 0$.

7.9 Annex – Case C2

Case C2 – Control Panel

Goldilocks Curves - CASE C1

Parabola and Parabola

1 For case C1, set c to 0, then copy D12 to D11.

Control Panel

I=Parabola		**C=Parabola**		**Constants**		**Intercepts at w_U=80**		**Slopes at w_U=80**	
$I = a(w-b)^2 + c$		$C = d(w-f)^2 + g$				α_I =	25925	β_I =	-295
a	-5	d	-3	e	2.718282	α_C =	18885	β_C =	-207
b	50	f	45	T	1	$(\alpha_I-\alpha_C)/\alpha_I$ =		$(\beta_I-\beta_C)/\beta_I$ =	
c	6825	g	6000				0.986784		0.298305
w=80 c parm	0			P_{max}	1012				

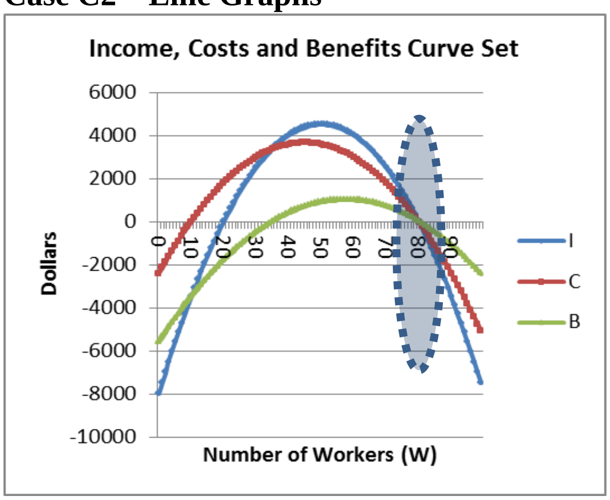
			B=I-C	P=B/T	η=B/I	ζ=P/P_{max}	ROI=B/C	ζ=P/P_{max}	η=B/I	ROI=B/C
W	**I**	**C**	**B**	**P**	**η**	**ζ**	**ROI**	**ζ**	**η**	**ROI**
0	-5675	-75	-5600	-5600	0.986784	-5.5336	74.66667	-5.5336	0.986784	74.66667
1	-5180	192	-5372	-5372	1.037066	-5.3083	-27.9792	-5.3083	1.037066	-27.9792
2	-4695	453	-5148	-5148	1.096486	-5.08696	-11.3642	-5.08696	1.096486	-11.3642
3	-4220	708	-4928	-4928	1.167773	-4.86957	-6.96045	-4.86957	1.167773	-6.96045

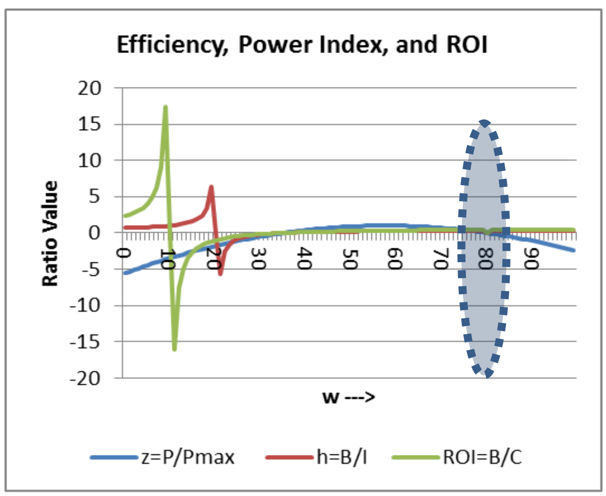
And, further down ...

W	**I**	**C**	**B**	**P**	**η**	**ζ**	**ROI**	**ζ**	**η**	**ROI**
76	1120	792	328	328	0.29286	0.32411	0.41414	0.32411	0.29286	0.41414
77	855	603	252	252	0.29474	0.24901	0.41791	0.24901	0.29474	0.41791
78	580	408	172	172	0.29655	0.16996	0.42157	0.16996	0.29655	0.42157
79	295	207	88	88	0.29831	0.08696	0.42512	0.08696	0.29831	0.42512
80	0	0	0	0	#DIV/0!	0	#DIV/0!	0	#DIV/0!	#DIV/0!
81	-305	-213	-92	-92	0.30164	-0.09091	0.43192	-0.09091	0.30164	0.43192
82	-620	-432	-188	-188	0.30323	-0.18577	0.43519	-0.18577	0.30323	0.43519
83	-945	-657	-288	-288	0.30476	-0.28458	0.43836	-0.28458	0.30476	0.43836

I have arbitrarily decided to use two parabolic curves to demonstrate this characteristic.

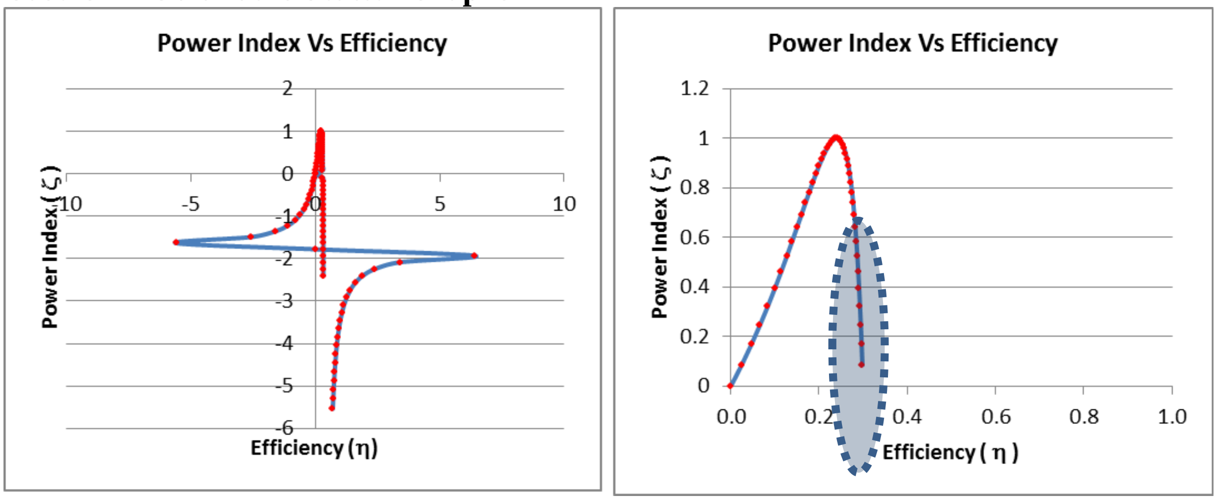
Case C2 – Line Graphs





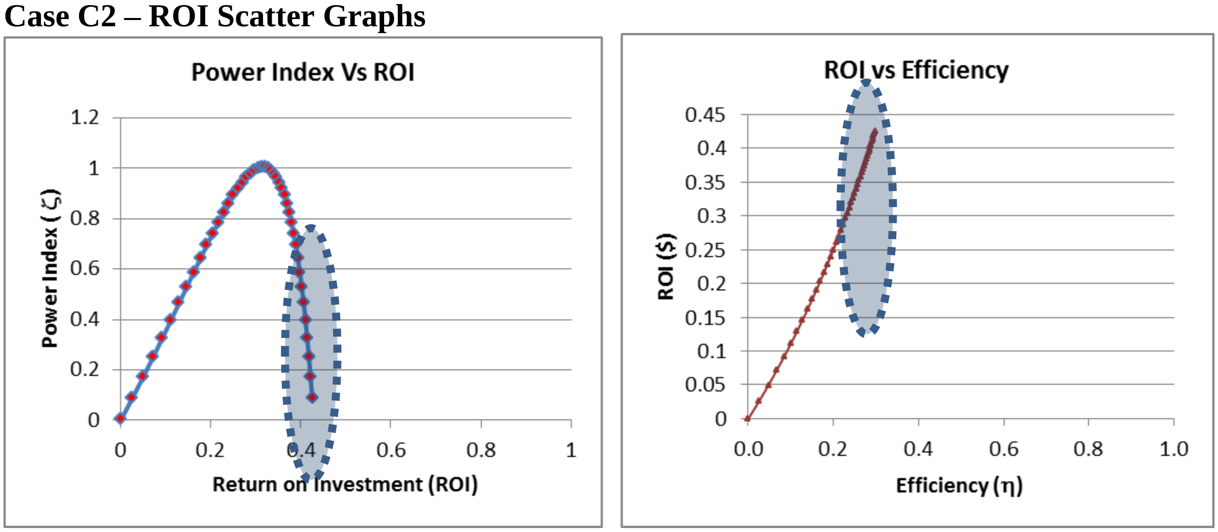
The curve set is an element of Π.
The focus is on w=w_U=80. I(w_U)=0. C(w_U)=0. B(w_U) = 0. w_U, the upper bound of the interval on which B(w) is positive, is located at w=80.

Case C2 – Goldilocks Scatter Graphs



Some of the data points are meaningless, falling elsewhere than in the 1st quadrant. The meaningful points are graphed to the right. The shaded oval indicates the area of interest in the graphs.
Focus on w = 80. $\zeta(w_U) = 0$. $\eta(w_U) = (\beta_I - \beta_C)/\beta_I$.

Case C2 – ROI Scatter Graphs



Only the 1st quadrant is included in the graphs.
Focus on w = 80. $\zeta(w_U) = 0$. $\eta(w_U) = (\beta_I - \beta_C)/\beta_I$. $R(w_U) = (\beta_I - \beta_C)/\beta_C = \eta/(1-\eta)$

7.10 Annex – Case D

Case D – Control Panel

Goldilocks Curves - CASE D

Attenuated Sine Curves

Control Panel

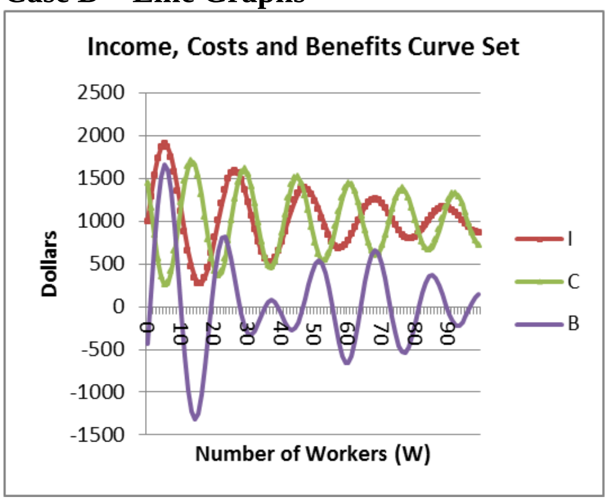
I=Attenuated Sine **C=Shiftable Attenuated Sine** **Constants**

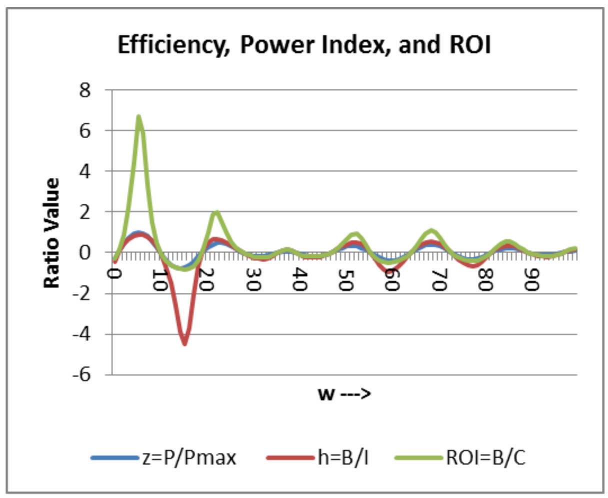
$I = ae^{bw}\sin(cw) + d$ $C = fe^{gw}\sin(hw - k) + m$

a	1000	f	800		e	2.718282				
b	-0.02	g	-0.01		T	1				
c	0.3	h	0.4							
d	1000	k	10		P_{max}	1655.456				
		m	1000	P=B/T	η=B/I	ζ=P/P_{max}	ROI=B/C	ζ=P/P_{max}	η=B/I	ROI=B/C
W	**I**	**C**	**B**	**P**	**η**	**ζ**	**ROI**	**ζ**	**η**	**ROI**
0	1000	1435.216889	-435.2169	-435.217	-0.43522	-0.2629	-0.30324	-0.2629	-0.43522	-0.30324
1	1289.669	1138.073761	151.59475	151.5948	0.117546	0.091573	0.133203	0.091573	0.117546	0.133203
2	1542.503	825.2188815	717.28364	717.2836	0.465013	0.433285	0.869204	0.433285	0.465013	0.869204

I have arbitrarily decided to use two sine curves, each within its own declining exponential curve acting as an envelope to the amplitude, to demonstrate this characteristic.

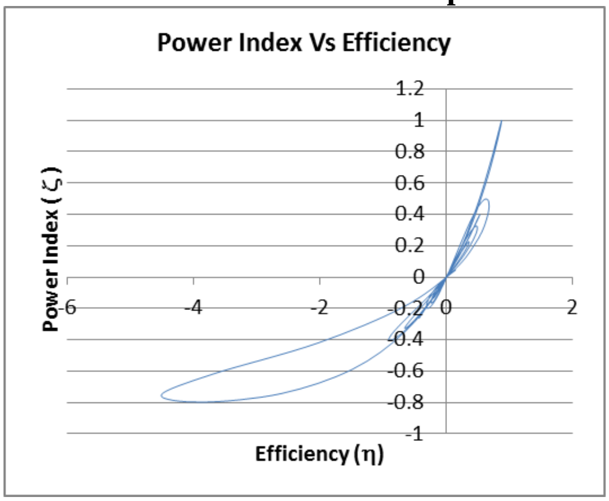
Case D – Line Graphs

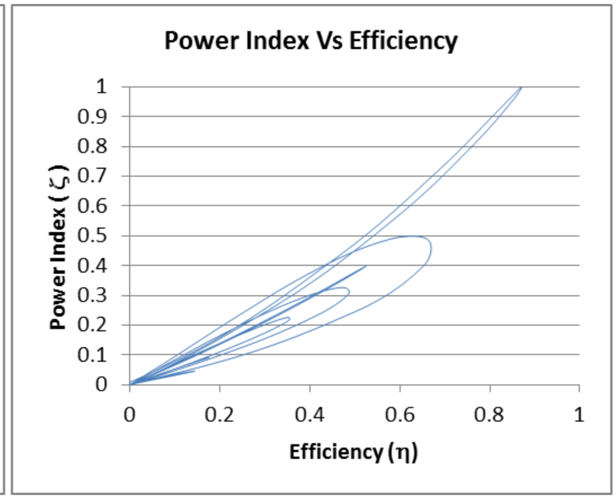




The curve set is an element of Π, but exhibiting recurring intervals of w over which B(w) is positive.
The focus is on those intervals over which B(w) is positive, of which there are too many to highlight.

Case D – Goldilocks Scatter Graphs

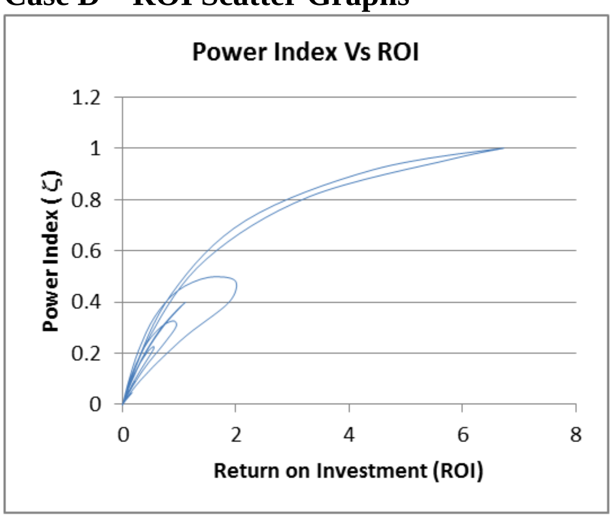


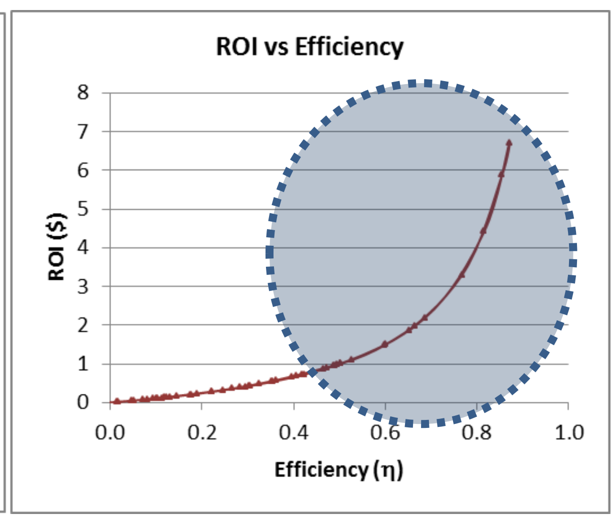


Some of the data points are meaningless, falling elsewhere than in the 1st quadrant. The meaningful points are graphed to the right.

No particular focus. Each interval of w over which B(w)>0 has produced its own Goldilocks looping curve. If this were to represent a business model, there is clearly one interval that provides both high power and high efficiency, and that is the interval w∈[1, 9] where benefits are maximized.

Case D – ROI Scatter Graphs





Only the 1st quadrant is included in the graphs.

In the left-hand graph, the point of maximal power coincides with the point of maximal ROI. In the right-hand graph ROI and efficiency seem to have a characteristic relationship for all loops. I.e. $R = -\eta/(1-\eta)$. Interesting! This is independent of w. It is, of course, obvious once you see it, but I did not see it until I looked at this graph closely. This seems to be true of ALL of these R vs η graphs for all cases. I have gone back in revision mode and noted this for each.

7.11 Annex – Case E1a

Case E1a – Control Panel

Goldilocks Curves - CASE E1a

Asymptotes

Control Panel

I=TanH() — $I = a + b\ tanh(c(w - d))$

C=TanH() — $C = f + g\ tanh(h(w - k))$

Constants

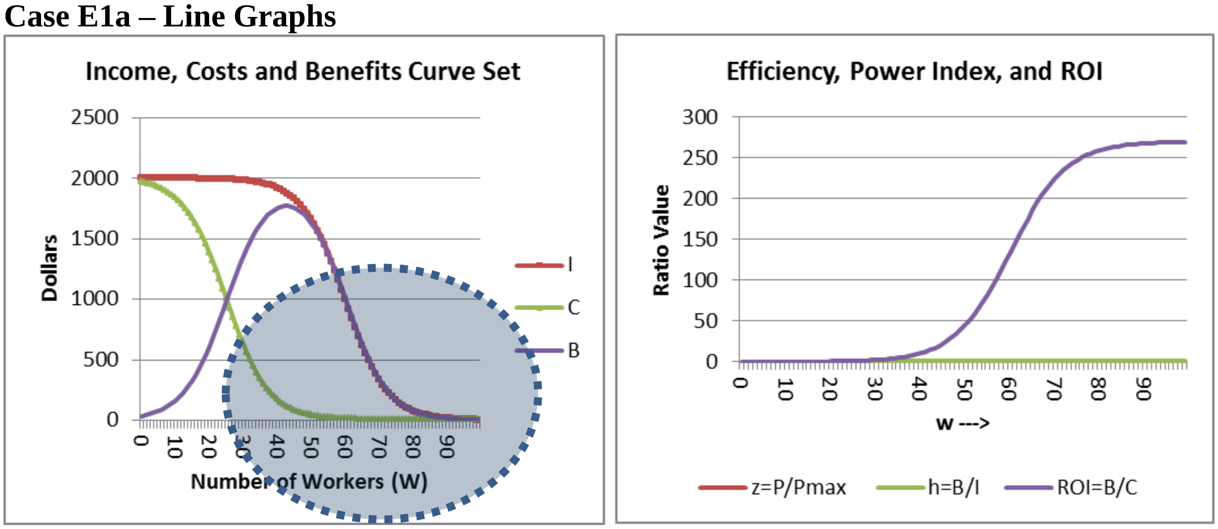
a	0	f	0
b	1000	g	1000
c	0.08	h	0.08
d	60	k	25

e	2.718282
T	1
P_{max}	1770.091

				P=B/T	η=B/I	ζ=P/P_{max}	ROI=B/C	ζ=P/P_{max}	η=B/I	ROI=B/C
W	**I**	**C**	**B**	**P**	**η**	**ζ**	**ROI**	**ζ**	**η**	**ROI**
0	1999.865	1964.02758	35.83697	35.83697	0.01792	0.020246	0.018247	0.020246	0.01792	0.018247
1	1999.841	1957.917306	41.92375	41.92375	0.020964	0.023685	0.021412	0.023685	0.020964	0.021412
2	1999.813	1950.795143	49.01833	49.01833	0.024511	0.027693	0.025127	0.027693	0.024511	0.025127
3	1999.781	1942.503008	57.27811	57.27811	0.028642	0.032359	0.029487	0.032359	0.028642	0.029487

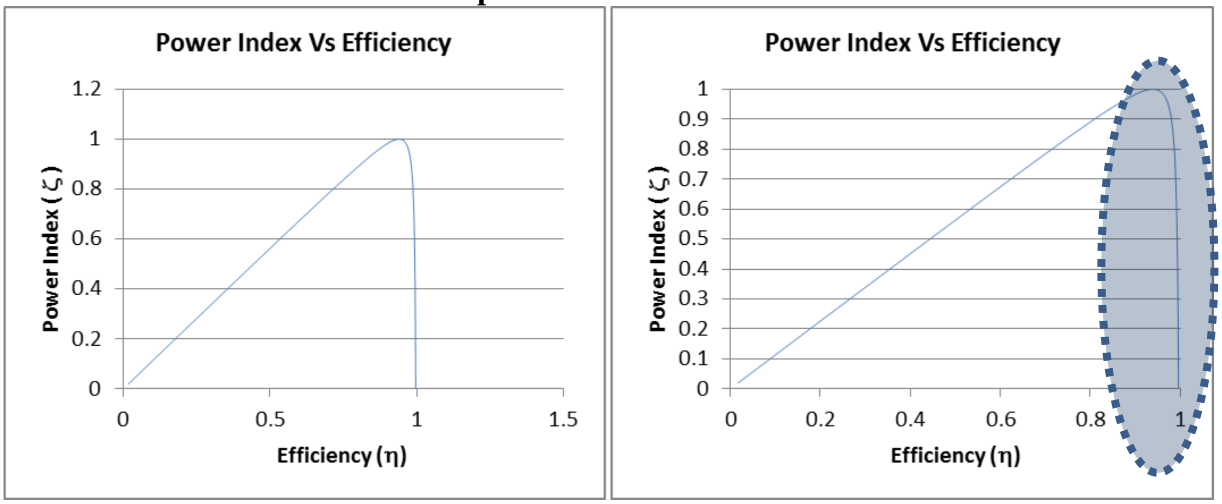
I have arbitrarily decided to use two hyperbolic tangent curves to demonstrate this characteristic.

Case E1a – Line Graphs



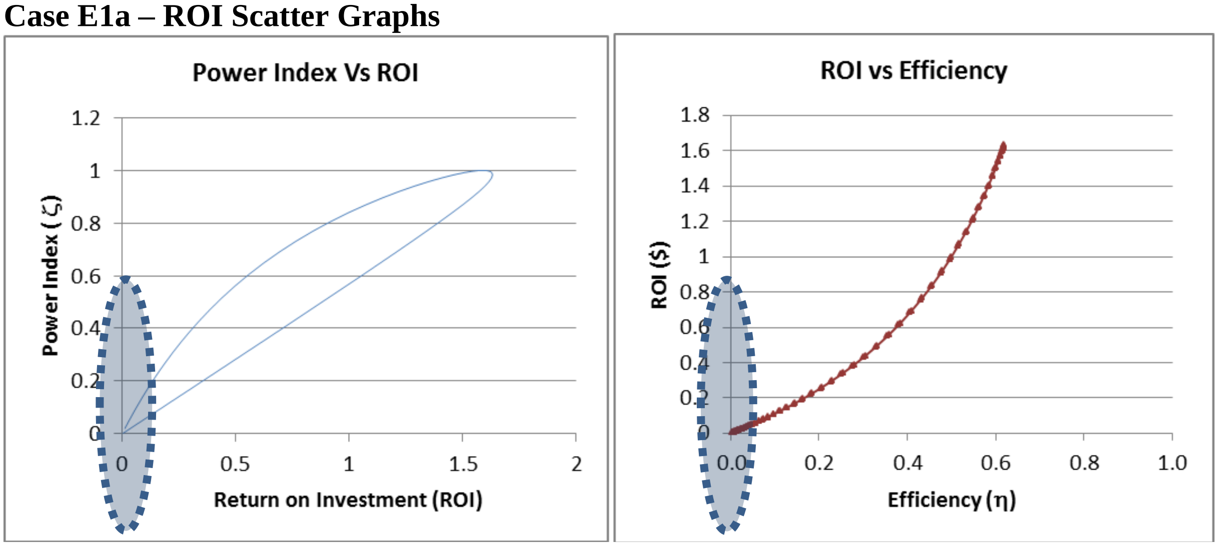
The curve set is an element of Π, but exhibiting a δ-neighbourhood of ∞ over which B(w) is positive and approaching zero**.** The focus is on this δ-neighbourhood of ∞.

Case E1a – Goldilocks Scatter Graphs



All of the data points are in the 1st quadrant, so both graphs look the same.
Focus on the δ-neighbourhood of ∞.

Case E1a – ROI Scatter Graphs



Only the 1st quadrant is included in the graphs.

7.12 Annex – Case E1b

Case E1b – Control Panel

Goldilocks Curves - CASE E1b

Asymptotes

Control Panel

I=TanH() $I = a + b\ tanh(c(w-d))$

C=TanH() $C = f + g\ tanh(h(w-k))$

Constants

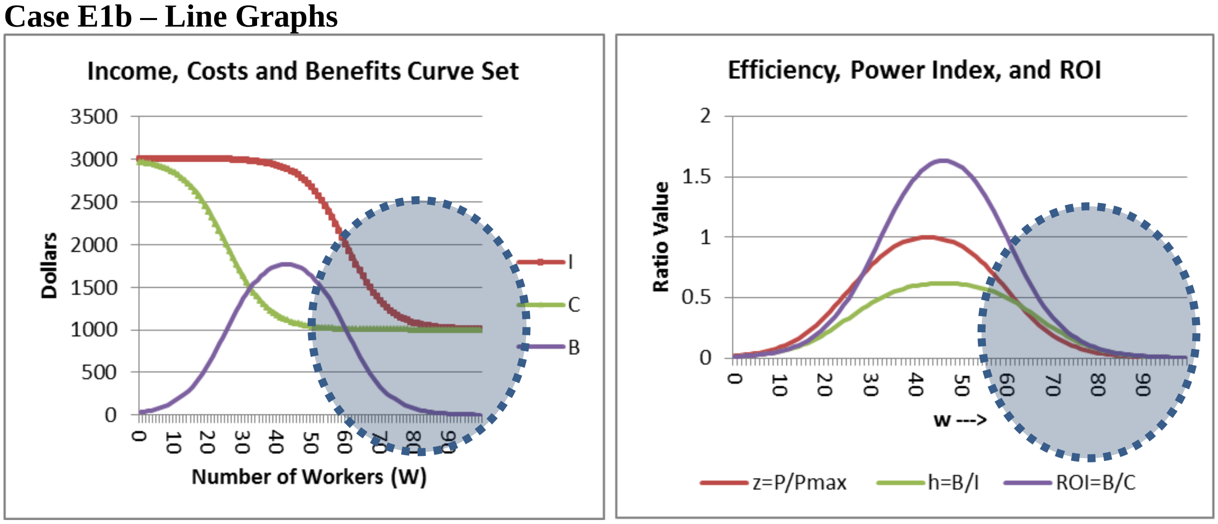
a	1000	f	1000
b	1000	g	1000
c	0.08	h	0.08
d	60	k	25

e	2.71828
T	1
P_{max}	1770.09

				P=B/T	η=B/I	ζ=P/P_{max}	ROI=B/C	ζ=P/P_{max}	η=B/I	ROI=B/C
W	**I**	**C**	**B**	**P**	**η**	**ζ**	**ROI**	**ζ**	**η**	**ROI**
0	2999.865	2964.02758	35.83697	35.837	0.01195	0.02025	0.01209	0.02025	0.01195	0.01209
1	2999.841	2957.91731	41.92375	41.9237	0.01398	0.02368	0.01417	0.02368	0.01398	0.01417
2	2999.813	2950.79514	49.01833	49.0183	0.01634	0.02769	0.01661	0.02769	0.01634	0.01661

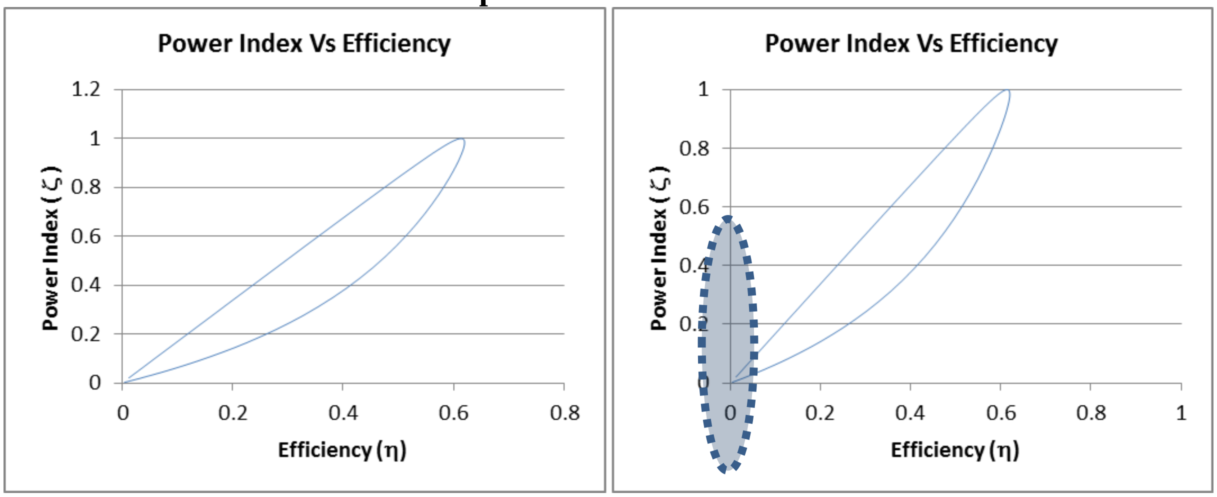
I have arbitrarily decided to use two hyperbolic tangent curves to demonstrate this characteristic.

Case E1b – Line Graphs



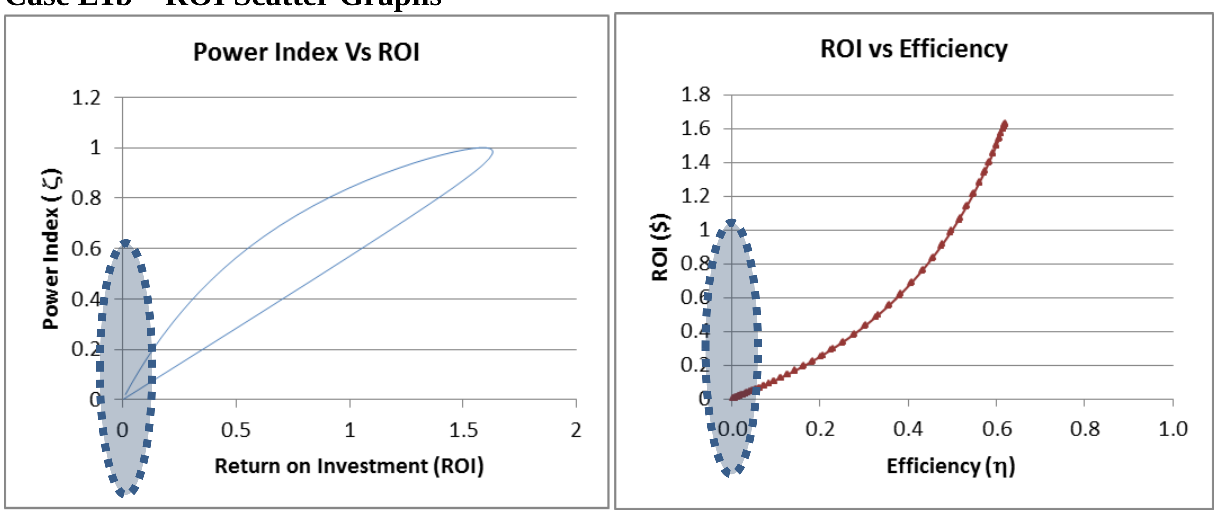
The curve set is an element of Π, but exhibiting a δ-nighbourhood of ∞ over which B(w) is positive, which is the focus.

Case E1b – Goldilocks Scatter Graphs



All of the data points are in the 1st quadrant, so both graphs look the same.
Focus on the δ-neighbourhood of ∞.

Case E1b – ROI Scatter Graphs



7.13 Annex – Case E1d

Case E1d – Control Panel

Goldilocks Curves - CASE E1d

Asymptotes

Control Panel

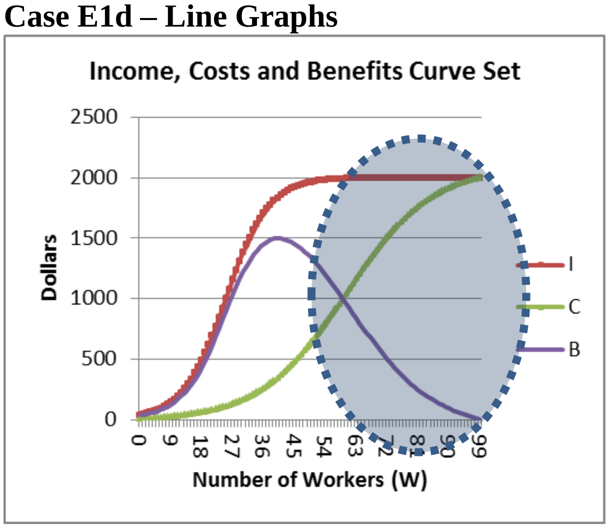
I=TanH() **C=TanH()** **Constants**

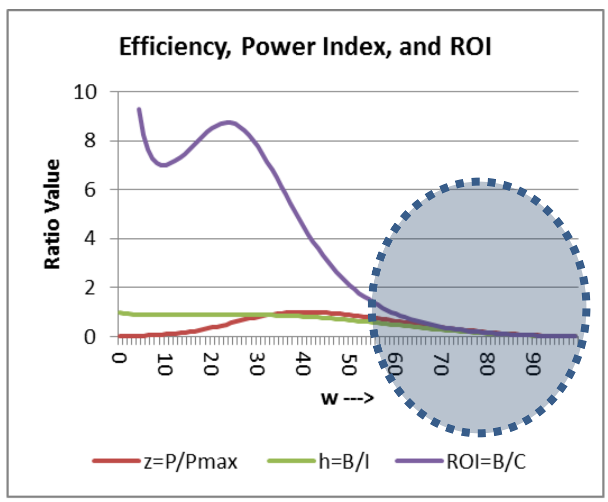
$I = a + b\ tanh(c(w - d))$ $C = f + g\ tanh(h(w - k))$

a	1000	f	1033		e	2.718282				
b	1000	g	1050		T	1				
c	0.08	h	0.04							
d	25	k	60		P_{max}	1497.893				
				P=B/T	η=B/I	$ζ=P/P_{max}$	ROI=B/C	$ζ=P/P_{max}$	η=B/I	ROI=B/C
W	**I**	**C**	**B**	**P**	**η**	**ζ**	**ROI**	**ζ**	**η**	**ROI**
0	35.97242	0.141399422	35.83102	35.83102	0.996069	0.023921	253.4029	0.023921	0.996069	253.4029
1	42.08269	1.556440967	40.52625	40.52625	0.963015	0.027056	26.03777	0.027056	0.963015	26.03777
2	49.20486	3.087169211	46.11769	46.11769	0.937259	0.030788	14.9385	0.030788	0.937259	14.9385

I have arbitrarily decided to use two hyperbolic tangent curves to demonstrate this characteristic.

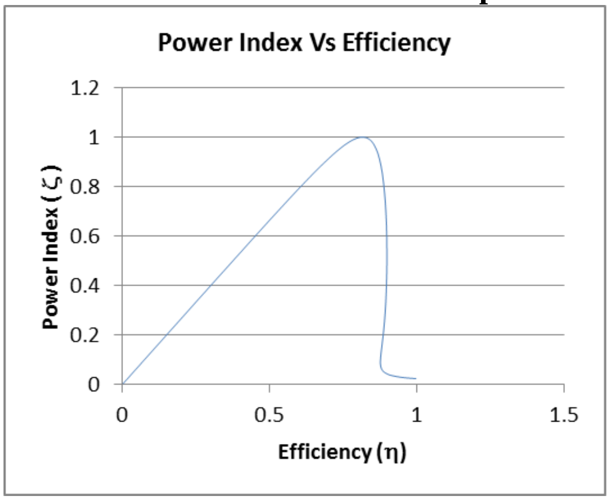
Case E1d – Line Graphs

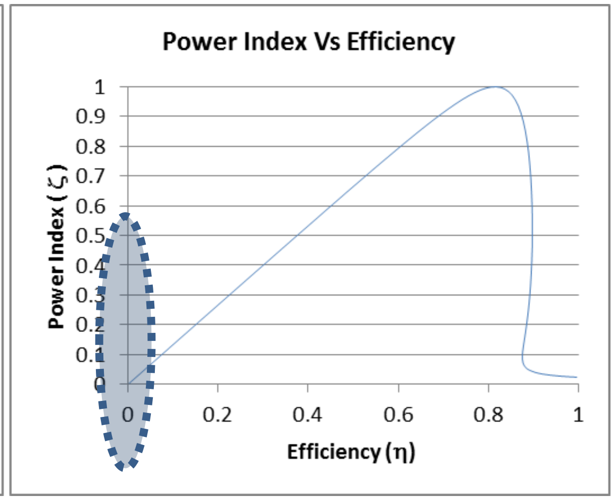




The curve set is an element of Π, but exhibiting a δ-nighbourhood of ∞ over which B(w) is positive, which is the focus.

Case E1d – Goldilocks Scatter Graphs





All of the data points are in the 1st quadrant, so both graphs look the same.
Focus on the δ-neighbourhood of ∞.

Case E1d – ROI Scatter Graphs