NOTE TO FILE
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Dated: 150105

Further Study of the shape of the AM curve

1 References

A. Email Boyle to Hall 150104
B. Email Hall to Boyle 150105
C. 150105 Atwood's Machine - Expanded Graphs R1.xlsx
D. 150101 NTF Atwood's Machine R4.docx
E.

2 Purpose

To do some further investigations of the shape of the curve that is characteristic of the output of power of Atwood's Machine.

3 Background

At Ref A I mentioned to Dr Hall that the curve was asymmetric. I was concerned that this might be an error. But he confirmed that it was/is asymmetric. He pointed out that in Odum's "Systems Ecology" book (1983) there was a figure (7-20), with associated mathematics, that showed it to be asymmetric.
Email – Boyle

4 Discussion

At Ref C I put together a variety of combinations of E (Efficiency) and P (Power) where I am examining the shape of the curve. The physical scaling constant is not needed to examine shape, and so has been set to 1. What remains of the equation describing the curve is:

$$P(E) = \sqrt[2]{\frac{E^2(1-E)}{(1+E)^3}}$$	Equ 01

This equation was derived at equation 20 of Ref D. In this note I distinguish between $P_S(E)$, the original AM curve at equation 20 of Ref D, P(E), which is just the variable part, as per equ 01 above, and $P_{index}(E)$, which is a scaled version of P(E) which I introduce below. Most of my graphs are of $P_{index}(E)$, but the other two can be recovered by multiplication by the appropriate scaling constants.

4.1 Wolfram Alpha

I fed this into Wolfram Alpha at Ref E and got a wealth of information.

E.g. The real domain is the interval [-1 < x <= 1]. This means it is defined for negative efficiencies. I doubt it has real meaning for negative efficiencies. This would correspond to an AM in which M_H would be lighter than M_L, and the machine would not function. Another way to say that is, the machine has reached the end of its useful cycle. Or, the machine is primed to work in reverse? As I say, no real meaning. Nevertheless, it has a VERY peculiar shape over the entire domain. (See Figure 01 below.) At x = -1 there is a vertical asymptote. At s=0 it looks like an absolute value function. At x=1 it looks like a collapsing function.

Wolfram alpha also provides a series expansion for each of x=-1, x=0, x=1, and x=∞. Really cool! But I don't see a use for them at this point.

Wolfram alpha also provides a derivative function and a definite integral. The derivative is of interest.

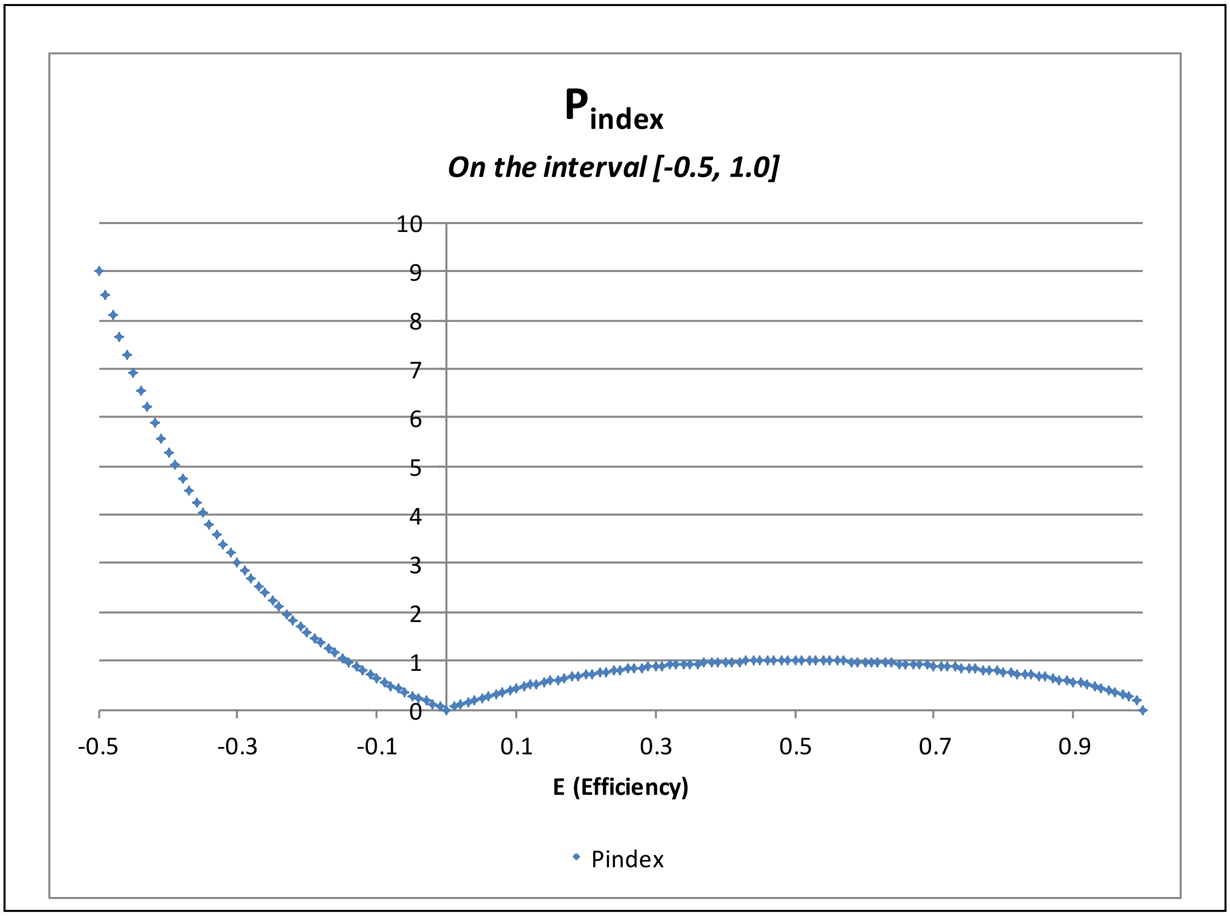
$$\frac{dP}{dE} = \frac{d}{dE}\left[\sqrt[2]{\frac{E^2(1-E)}{(1+E)^3}}\right] = \frac{E - 2E^2}{\sqrt[2]{\frac{E^2(1-E)}{(1+E)^3}}(1+E)^4}$$	Equ 02

Setting this equal to zero and solving for E will tell me the point where P(E) is at a maximum value. That's not an easy thing to do analytically.

Thankfully, Wolfram Alpha also provides the answer for that. A local maximum occurs at x = ½ and P(½)=1/(3*root(3))=0.19245. This concurs with Dr Hall's statement that maximum power occurs when x = ½.

Using this information, and, basically, just playing with various graphs as questions arose, I produced seven graphs. I reproduce them here one per page.

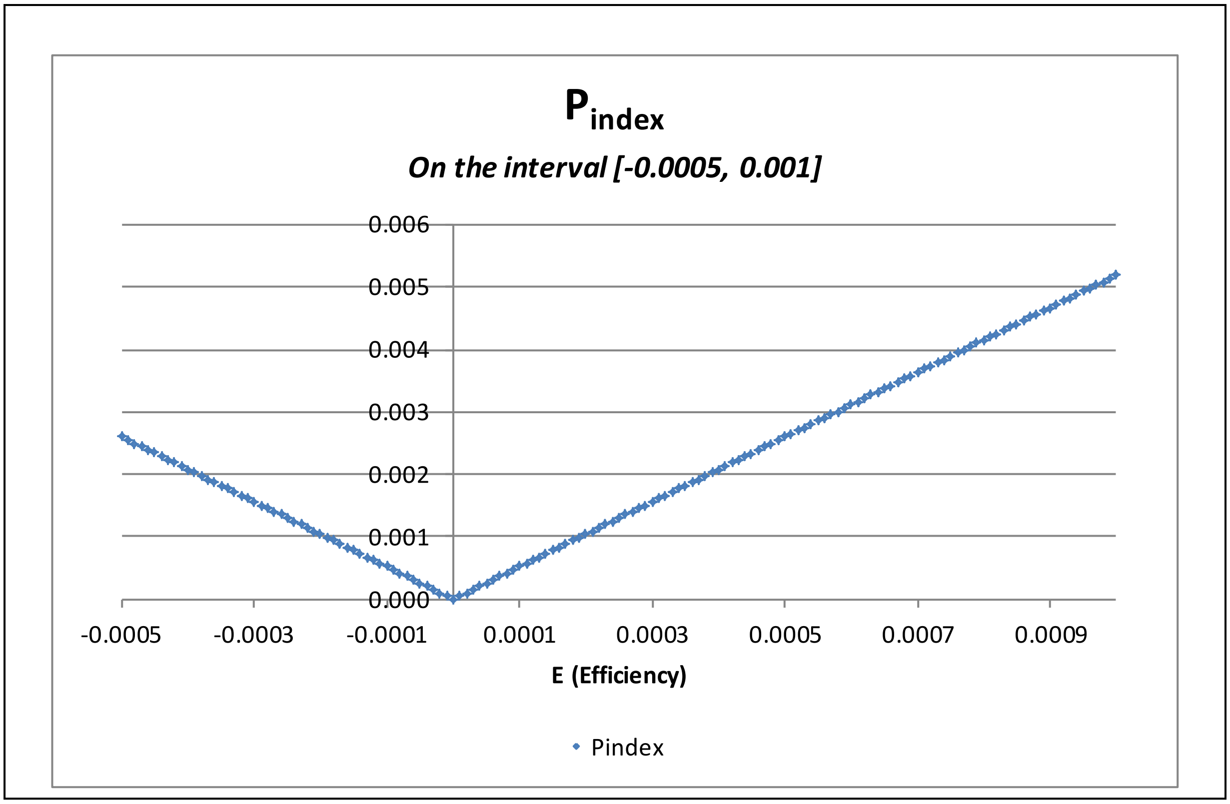
4.2 Figure 01 – P vs E – [-0.5, 1.0]



This is not actually the curve P(E). Instead, it is the curve $P_{index}=P(E)/P_{max}$ where P_{max} is the local maximum of P(E) on the interval [0, 1].

It seems that the asymmetry of the portion of interest, on the interval [0, 1], derives from the strange behaviour at x=0.

4.3 Figure 02 – P vs E – [-0.0005, 0.001]

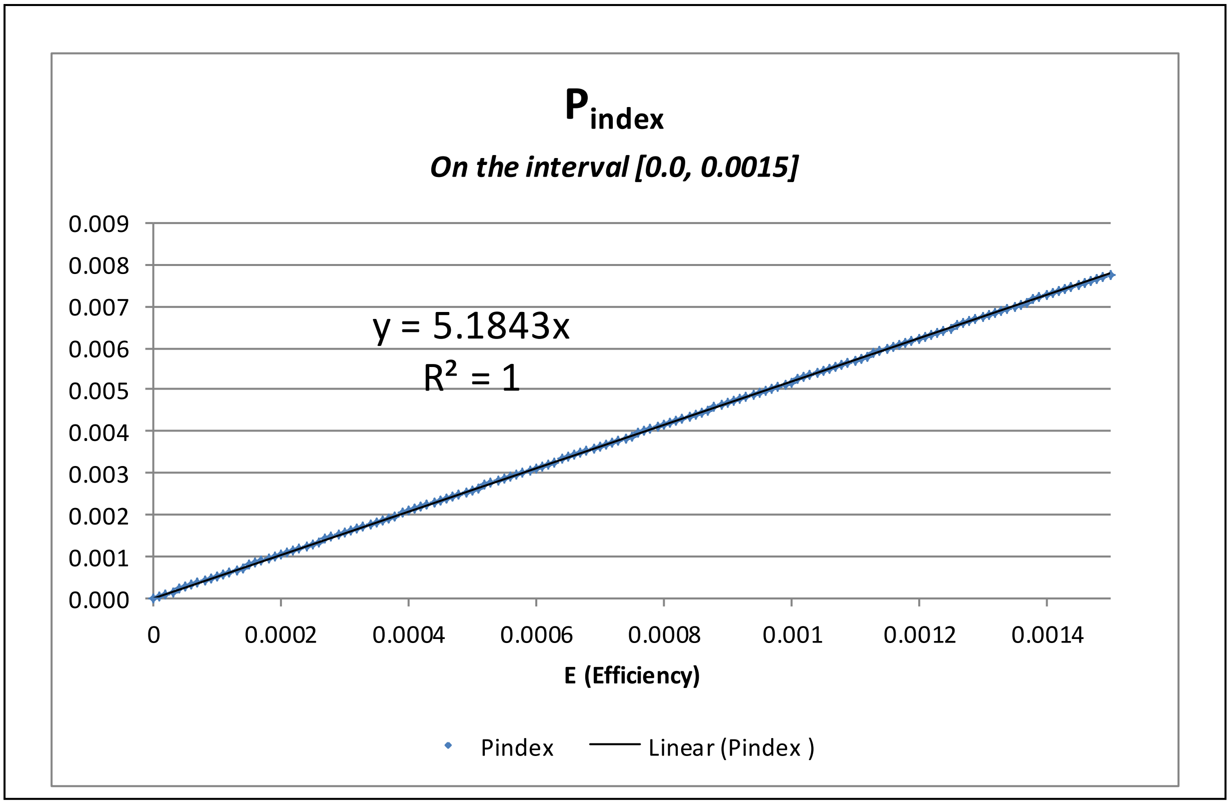


This is a close-up of the curve $P_{index}=P(E)/P_{max}$ on the interval [-0.0005, 0.001].

I focused on the spot x=0 going closer and closer but it is always similar. This looks like an absolute value function, with lines of similar slope but opposite sign. This makes sense, because it is a square root function, which would have only positive real values. I am ignoring the complex part of the graph.

So, the AM curve is linear on the left, near E=0. I wonder what the slope is and whether it has significance. In the next graph I look at the slope.

4.4 Figure 03 – P vs E – [0., 0.0015]



With lesser focus the R^2 value was less than 1. I kept zooming in until the R^2 was 1.

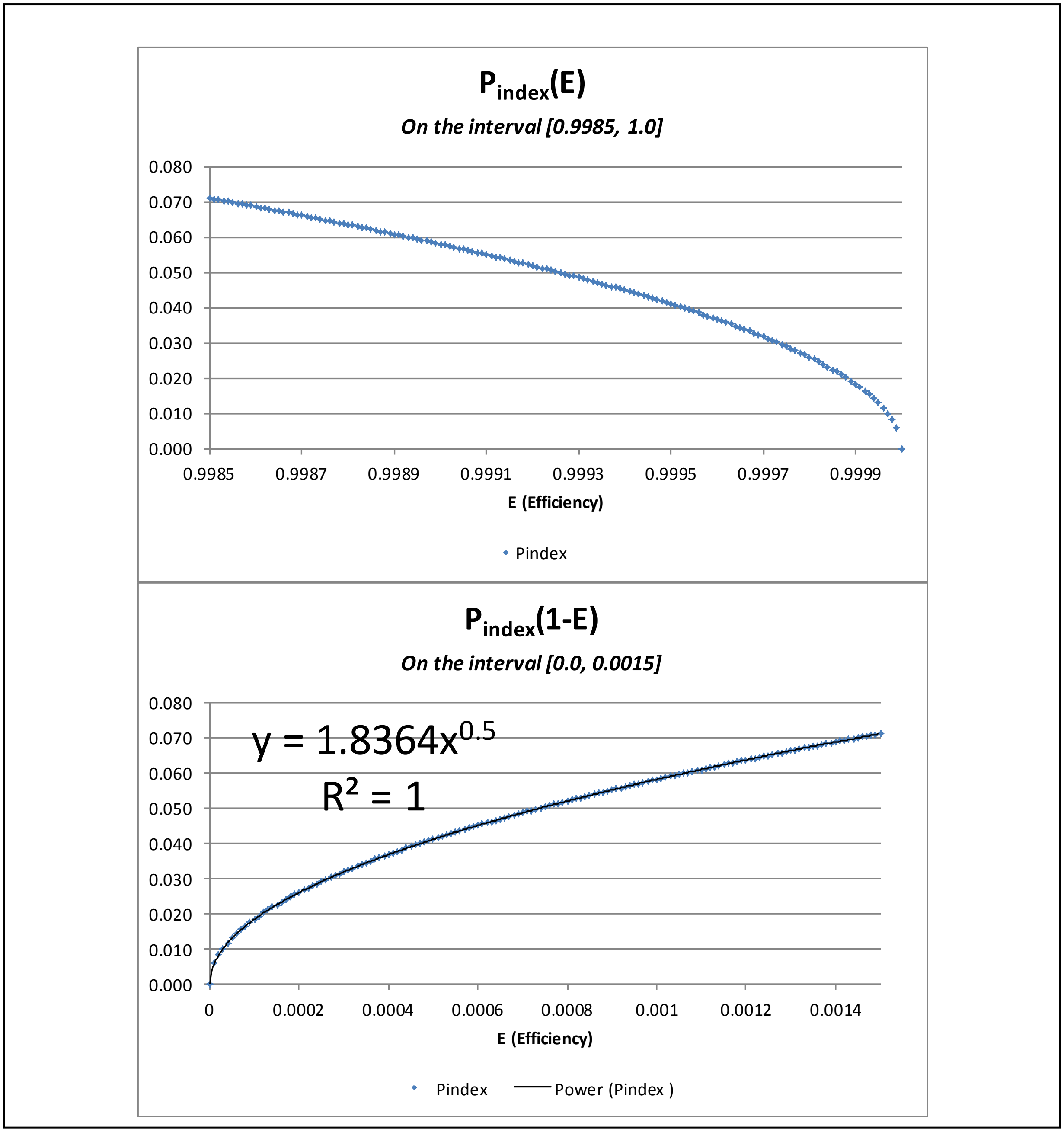
Curiously, the slope of $P_{index}(E)$, when E is very close to zero, is 5.1843. $P_{index}=P(E)/P_{max}$ where P_{max} is the local maximum of P(E) on the interval [0, 1].

The instantaneous slope of P(E) when E is very close to zero is then 5.1842 / 3Root(3) = 0.997719 fractional units of power increase per fractional unit of efficiency increase, and its linear. To convert this to the slope of $P_S(E)$ for the AM I would need to multiply it by the physical scaling constant C that I left out of this exercise. From equation 20 of Ref D, the physical scaling constant is:

$$C = \sqrt[2]{\frac{Dg^3 {M_t}^2}{2}}$$	Equ 03

I wont do that multiplication. The real question is, what is this number 5.1842? What is its significance?

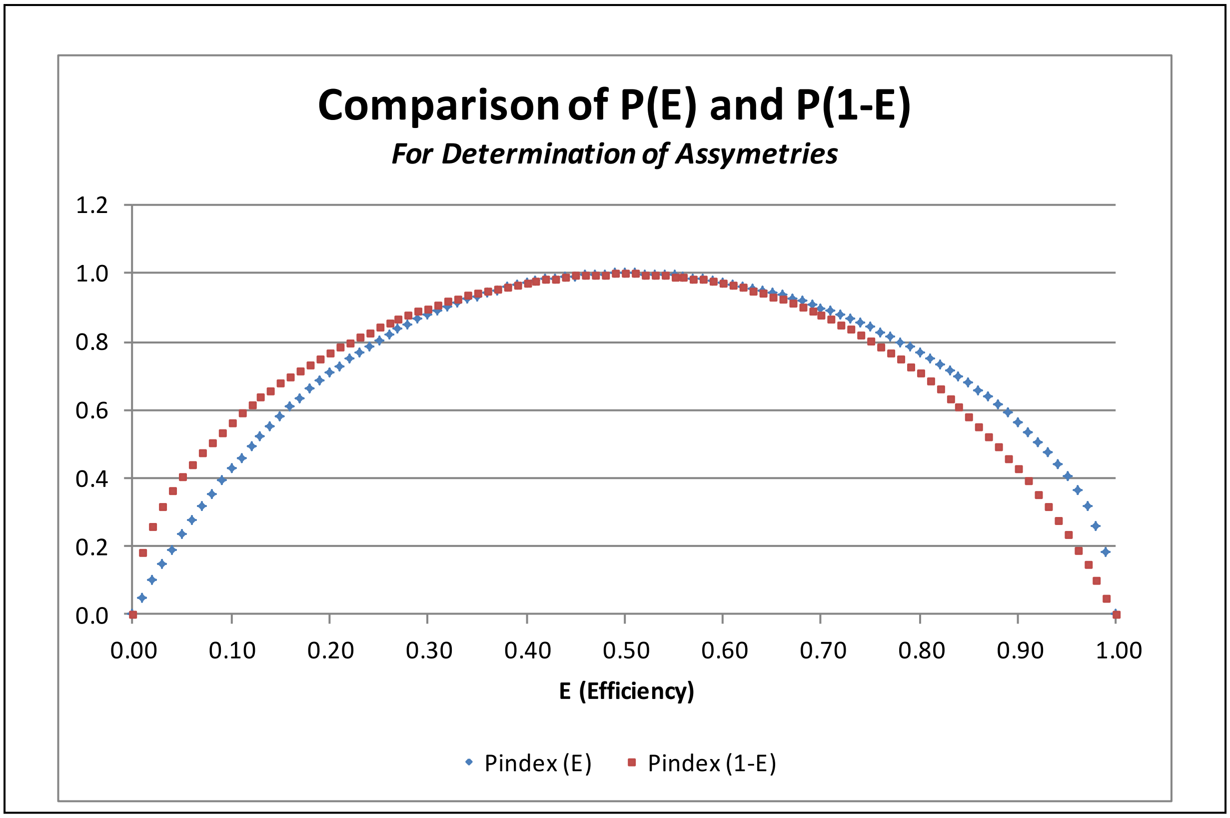
4.5 Figures 04 and 05 – P vs E – [0.9985, 1.0]



Looking at the other end of the [0, 1] interval, the $P_{index}(E)$ curve has a peculiar shape here, which is definitely not linear, even after I have zoomed in this far. It looks vaguely square rootish on a variable G=(1-E). So I did a sub, and got this second outstanding graph, showing it to be precicely square rootish.

My first conclusion, here, is this is definitely NOT a symmetric curve. BUT, what a strange sort of asymmetry. And what is the meaning of this number 1.8364?

4.6 Figure 06 – P vs E – [0, 1]



This was just so I could ‘eyeball’ the differences in symmetry.

5 Conclusion

For reasons that seemed good at the time that I did it, for this exercise I have worked with three closely related versions of the AM curve. I defined three related functions as follows:

- The characteristic curve of power (P) vs efficiency (E) for the AM was called $P_S(E)$; At the Ref D document I called this curve P_{AM} (the one that I produced indirectly using a scatter plot) and $P_S(E)$ which was derived from first principles at equation 20 of Ref D. This function has two parts: a physical constant C which is determined by the sizes of the masses and the distances travelled, and a variable part that is dependent on E.
- A scale-less version, where the physical constant C has been divided out, defined as $P(E) \equiv P_S(E)/C$;
- An indexed version, where it has been divided by its local maximum P_{max}, to make all graphs top out at 1.00, as $P_{index}(E) = P(E) / P_{max} = P_S(E) / (C * P_{max})$.

Or, $P_S(E) \equiv C * P(E) \equiv P_{max} * C * P_{index}(E)$

The AM curve is definitely strange. It has some hidden constants that are not affected by variations in the physical setup. It has strange behaviour at the ends of the domain. If the MPP is valid, it forms an attractor at it maximal point. Here is my list of odd facts:

- It is asymmetric on the interval of interest, i.e. $E \in [0, 1]$;
- P(E) has a maximal value at E = ½ where $P_{max} = 1 / (3 * \text{root}(3)) = 0.19245$;
- This means that $P_s(E)$ has a maximal value of $(P_{max} * C)$ when E = ½; This is somehow a key value, as all systems migrate towards this maximal value as they self-organize, according to H. T. Odum.
- All three functions are linear when E is very close to zero:
  - the slope of $P_{index}(E)$ is 5.1843;
  - the slope of P(E) is $5.1843 * P_{max} = 0.997719$;
  - the slope of $P_S(E)$ is 0.997719 * C;
- What is this hidden constant 0.997719?
- All three functions are power curves when E is very close to 1, with a power of 0.5 and a parameter 1.8364:
  - $P_{index}(E) \approx 1.8364 (1-E)^{½}$
  - $P(E) \approx P_{max} * 1.8364 (1-E)^{½} \approx 0.353415 (1-E)^{½}$
  - $P_s(E) \approx C * P_{max} * 1.8364 (1-E)^{½} \approx C * 0.353415 (1-E)^{½}$
- What is this hidden constant 0.353415?

I have looked up both of these ‘hidden’ constants on Wolfram Alpha and got nothing.

Hmm?